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Shear Deformations of FGM Beams with Mixed Restraints under Bending Using HSD Theory

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ABSTRACT

This paper presents a new proposed mathematical model for functionally graded material (FGM) beams with mixed restraints. Higher order shear deformation theory (HSDT) is used for FGM prismatic beams under bending taking into account the shear strains in the displacement field. A novel polynomial shear function is utilized for the shear stress which implies the nullity at the both top and bottom fibers faces of the cross-section, and its maximal value at the midline fiber. Material Characteristics are assumed to be varied continuously through the total thickness following a simple power-law, according to the volume fractions between constituents. The virtual work principle is employed for deriving the governing equations of the beam solved analytically using the integrals to obtain the displacement and the constitutive stress-strain relations with efficacy deterministic manner. It is noted that the material of FGM beam obeys Hooke's law. The equilibrium equations and boundary conditions are considered in this study. Furthermore, the novel analytical model is explored using two illustrative examples. The obtained results predicted from the new proposed model for clamped-simply supported (C-S) beams under bending, are in good agreement with those obtained from the available literature.

KEYWORDS

FGM beams
Mixed boundary conditions
Bending
HSD Theory
Displacement
Shear Stress

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1 Introduction

The effect of transverse shear in the theory of composite beams in functionally gradient materials generally describes the variation of materials profile as a function of its thickness. The classical Euler–Bernoulli (CT) beam theory was studied without considering shear deformation and has limitations for thick beams. Timoshenko et al. 1921 developed the first beam theory with shear deformation (FSDBT) applicable to bars, plates, shells, with Cartesian, polar, cylindrical, and spherical axes used deterministically.

This theory, which includes cross-section deformation (with a linear shear deformation effect), and uses a shear correction factor, makes it possible to distinguish the actual stress state from the assumed constant stress state [1]. Reddy et al. [2, 3] proposed a new theoretical formulation based on third order theory to analyze FGM plates. A comparison study using different shear deformation theories (SDT) (Parabolic PSDT, Trigonometric TSDT, and Exponential ESDT) was carried out to predict displacements and stresses of a rectangular plate with simply supported edges submitted to the bending, buckling, and free vibration, respectively [4]. As seen in Ghugal et al. works [5] exhibiting analytical solutions for thick isotropic beams with rectangular cross-section using different boundaries and loading conditions, the authors developed a shear deformation theory taking into account transverse shear deformation effects. Illustrative cases in terms of the displacement field were presented and compared to those of Timoshenko and other solutions. This comparison showed a best concordance. Some authors research works in the recent decades [6-29] were interested to study theoretically and experimentally the isotropic functionally graded thin and thick beams, FG sandwich beam, and plates with static and dynamic analysis under bending, buckling, and free vibration using beam theories (FSDT, HSDT,TSDT, ESDT, quasi-3D), or other theories, including the shear deformation effects across the thickness, also shed light ,according to several power law functions (linear, quadratic, cubic, inverse quadratic) taking into account the influence of material composition on bending analysis of functionally graded for short, thick FGM beams , boundary conditions, and the uniformly and non-uniformly loading, were investigated. Comparative and parametric analyses were conducted to influences material composition on the FGM structural response. In this subject, models were developed by differential equations obtained from the virtual work principle, Hamilton's principle, and a solution chosen for Navier-type analysis, or analysis also by other methods, to explore the displacement field, normal stress, shear stress, frequencies, and other parameters were studied. Results predicted from these models were compared to those found in previous

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researches, available in the literature, to verify their validity and the correctness and accuracy effects studied. In the reference [18], authors studied the non-linearized bending behaviors of a two-dimensionally functionally graded (FG) beam based on the Euler–Bernoulli beam kinematic theory. Therefore, the power-law distribution function is used to describe the variation of Young's modulus in beam's thickness direction. Furthermore, the exponential function is chosen to describe the variation of Young's modulus in beam's length direction. Otherwise, the Young's modulus varying along the length or axial direction obeys an exponential distribute function, and the Young's modulus varying along the thickness direction obeys a power-law function. Authors of the referenced works [9, 19, 17] developed new models, using elasticity solutions for bending, buckling, and free vibration analysis of thin and thick beams with different ends conditions, across the thickness, to obtain displacements, stresses, axial normal and transverse shear stresses with deterministic manner. In referenced research works [30, 31], authors proved that the transverse shear should be considered to predict the critical loads in FGM beam using finite element formulations and beams theories (FSDT and HSDT) for buckling and bending analysis of FGM beam based on the both Euler-Bernoulli theory and Timoshenko beam theory, respectively. Over the last two decades, the researchers focused their efforts on developing FGM structures to give efficient results, particularly, on the bending, buckling and vibrations. Although their work has still not been sufficient, the rise of sciences in global need more investment about FGM structures studies, and has become of increasing interest. For this reason, a new study is carried out about FGM beams with mixed restraints taking into account the shear strains in the displacement field. However, the previous research reviewed did not sufficiently study the behavior of FGM beams subjected to mixed restraints, under flexure. Several research works were carried out to investigate the behavior of FGM structures, it was observed that there is no relationship between Poisson's ratio and deflections, stresses distribution and strain energy. Furthermore, investigations on the static analyses of the FGM beams revealed that deflections, stresses and the location of the neutral surface are highly dependent on the power law index, and they are not dependent on Poisson ratio [32].

Several research works were carried out to investigate the behaviour of FGM structures. Although, their work has still not been sufficient, the rise of sciences in global need more investment about FGM structures studies, and has become of increasing interest. For this reason, a new study is carried out about FGM beams with mixed restraints taking into account the shear strains in the displacement field. However, the previous research reviewed did not sufficiently study the behaviour of FGM beams subjected to mixed restraints, under flexure.

In this paper, the main purpose is to develop a new exhaustive mathematical model for FGM beams with mixed restraints under bending using higher order shear deformation theory (HSDT). This novel contribution considered was not used elsewhere. The following text used only for thesis please separate the introduction into paragraphs and rewrite the following text as the aim of the work.

2 Beam materials characteristics

Figure 1 presents a FGM beam with geometric properties such as the length L, width b and thickness h. It is noted that the beam material properties vary continuously in the z- direction.

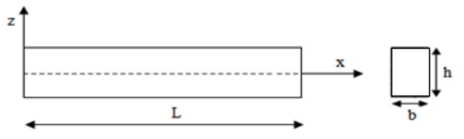

Fig.1. Geometric dimensions of beam.

The material properties $M_P(z)$ of mixing rule is given by:

$$M_p(z) = M_p V_c + M_m(1 - V_c) \quad (1)$$

The stiffness coefficients have been obeyed to the mixing rule of constituents E(z) and G(z), respectively (Appendix-1). The effective Poisson's ratio is held constant equal to 0.3 for this study. The following equations (2-a) and (2-b) depict the volume fractions of constituents:

$$V_m = 1 - V_c \quad (2\text{-a})$$

$$V_c = (\frac{z}{h} + \frac{1}{2})^P \quad (2\text{-b})$$

Rewrite the following sentence where, are defined ceramic volume fraction (V_c) and metal volume fraction (V_m).

The typical variation of the volume fraction of the constituents versus the thickness is drawn in Figure 2.

The neutral axis position over the thickness of the beam is given by the following equation:

$$h_0 = \frac{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) z dz}{\int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) dz} \quad (3)$$

It should be noted that the material properties of the studied prismatic beam are mentioned in the Appendix-1. Also, for the behavior of used materials, both the thickness (vertical) and the span length (axial) respect the same rule characteristics mentioned.

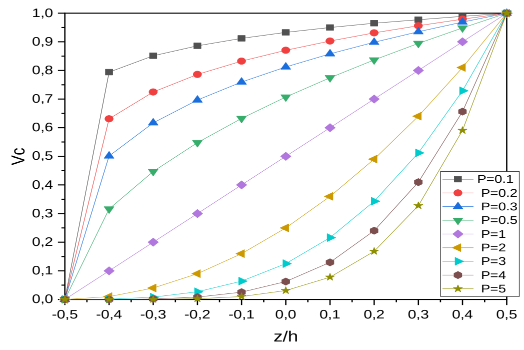

Fig. 2. Distribution of volume fraction V_c versus thickness (z/h).

3 Mathematical model

3.1 Boundary conditions

The essential boundary conditions (Neumann) and natural (Dirichlet) in terms of the displacement, rotation and the symmetry consideration, for the considered cases such astheclamped-simply supported (C-S) prismatic beam having a rectangular section (*bxh*) subjected to concentrated load P and uniformly distributed load q (Figure 3), are given by: $w = 0$ for $x = 0; x = L(4);$

$$D^*_{11}\frac{\partial^3 w_0}{\partial x^3} = D^*_{11}\frac{\partial^2 \varphi_x}{\partial x^2} = \frac{\partial w(x)}{\partial x} = \varphi_x = 0 \; for \; x = L/2 \quad (4\text{-a})$$

The corresponding boundary conditions of the present study are depicted in Table1. Where w_0, w_{0x} represent the displacement and their first derivative adopted, respectively. φ_x: is defined as the imposed deformed.

Table 1. Boundary conditions.

Boundary conditions	Left side at (x=0)	Right side at (x=L)
Simply supported (S-S)	$w_0=0, u_0 \neq 0, w_{0x} \neq 0, \varphi_x \neq 0$	$w_0=0, u_0 \neq 0, w_{0x} \neq 0, \varphi_x \neq 0$
Clamped-Clamped (C-C)	$w_0=0, u_0=0, w_{0x}=0, \varphi_x=0$	$w_0=0, u_0=0, w_{0x}=0, \varphi_x=0$
Clamped- Simply (C-S)	$w_0=0, u_0=0, w_{0x}=0, \varphi_x=0$	$w_0=0, u_0 \neq 0, w_{0x} \neq 0, \varphi_x \neq 0$
Clamped-Free (C-F)	$w_0=0, u_0=0, w_{0x}=0, \varphi_x=0$	$w_0 \neq 0, u_0 \neq 0, w_{0x} \neq 0, \varphi_x \neq 0$

3.2 Governing differential equations

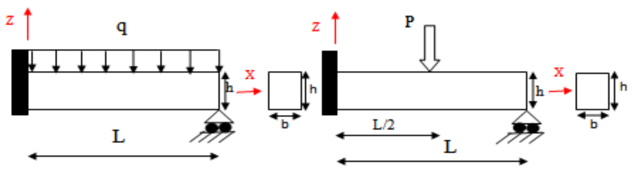

Fig.3. Clamped–simply supported (C-S) beam subjected to uniformly distributed load q and concentrated load P having a rectangular cross section (b x h).

The virtual work principle applied, in brief, on the beam (Figure 3) is defined as follows:

$$b\int_{x=0}^{x=L}\int_{Z=\frac{-h}{2}}^{Z=\frac{h}{2}}(\sigma_x\delta\varepsilon_x+\tau_{xz}\delta\gamma_{xz})dxdz-\int_{x=0}^{x=L}q\delta w dx=0 \quad (5)$$

The differential equations (6) and (6-a) obtained are as follows: Please note that you have used (2-a) and (2-b), but later you refer to (6) and (6-a). Kindly ensure consistency and logical numbering of the equations, both in their labeling and in the corresponding references within the text.

$$\frac{\partial^2\varphi_x}{\partial x^2}-\frac{D^*_{11}}{F^*_{55}}\left(\varphi_x+\frac{\partial w(x)}{\partial x}\right)=0 \quad (6)$$

$$\frac{\partial^2 w(x)}{\partial x^2}+\frac{\partial\varphi_x}{\partial x}+F^*_{55}q=0 \quad (6\text{-a})$$

3.3 Presentation of the displacement and the new shear function

The displacement components are defined by the following equations:

$$u(x,z)=u_0(x)-z\frac{\partial w(x)}{\partial x}+\varphi(z)(\frac{\partial w(x)}{\partial x}+\theta(x)) \quad (7)$$

$$V(x,z)=0 \quad (8)$$

$$w(x,z)=w_0(x) \quad (9)$$

Where u_0, w_0: Axial and the transverse displacements, respectively; $\varphi(z)[\partial w(x)/\partial x+\theta(x)]$: the deformation function. For the present study, the new non-linear shear function adopted is given by:

$$\varphi(z)=z\left(\frac{3}{8}h^2-\frac{27}{28}\right)-z^3\left(\frac{1}{2}-\frac{9}{7h^2}\right) \quad (10)$$

The first derivative of the above equation $\frac{\partial\varphi(z)}{\partial z}=0$ at both upper and lower fibers where $z=\pm\frac{h}{2}$, and consequently, the nullity of the shear stress. While, the second derivative equal to zero ($\frac{\partial^2\varphi(z)}{\partial z^2}=0$) implies that the shear stress is maximal at the midline fiber (z=0). The shear coefficient S is defined as follows:

$$S=\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2 \quad (11)$$

Only a pure flexure is considered, the novel transverse displacement of C-S beam subjected to concentrated load P at the middle (Case-1) is predicted by the following equation at x = L/2:

$$w_{C1}\left(x=\frac{L}{2}\right)=-\frac{PL^3}{4E_xbh^3}\left[\frac{5}{8}+3\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2\right] \quad (12)$$

While, the novel transverse displacement of C-S beam under uniformly distributed load q over the long span (Case-2) is evaluated at x = L/2 by the following expression:

$$w_{C2}\left(x=\frac{L}{2}\right)=-\frac{qL^4}{4E_xbh^3}\left[\frac{3}{8}+3\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2\right] \quad (13)$$

It is noted that the transverse displacements versus x of the above cases are given in Appendix-2.

4 Strain components

The normal strain $\varepsilon_x(x, z)$ and the shear deformation $\gamma_{xz}(x, z)$ are respectively given by:

$$\varepsilon_x(x,z)=\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+\left[z\left(\frac{3}{8}h^2-\frac{27}{28}\right)-z^3\left(\frac{1}{2}-\frac{9}{7h^2}\right)\right](\frac{\partial\theta}{\partial x}+\frac{\partial^2 w}{\partial x^2}) \quad (14)$$

$$\gamma_{xz}(x,z)=[\left(\frac{3}{8}h^2-\frac{27}{28}\right)-z^2\left(\frac{3}{2}-\frac{27}{7h^2}\right)]\psi_x \quad (15)$$

At the mid-line, the warping shear function $\varphi(z)\psi_x$ of the beam, for z = 0, including the rotation angle of cross-section perpendicular to the mid-line of the form $\theta=\partial u/\partial z$, is presented as follows: $\varphi(z)\psi_x=\varphi(z)(\theta+\partial w_0/\partial x)$. While, the normal stress and the shear stress are respectively given by:

$$\sigma_{xx}(x)=E(z)\varepsilon_x \quad (16)$$

$$\tau_{xz}(x)=G(z)\gamma_{xz} \quad (17)$$

5 Findings comparison and discussions

5.1 Comparison of shear functions

The new proposed shear function (Eq.10) is compared to Reddy's shear function given by:

$$\varphi(z)=z(1-\frac{4}{3}\frac{z^2}{h^2}) \quad (18)$$

It can be observed that the Reddy's PSDBT function is suggested for real solution ($\Delta=0$).

Furthermore, the both shear functions are in good agreement for z=±h/2. The new shear function exhibits a best convergence with that found by Eq. (18), satisfying the boundary conditions without stresses on the top and bottom surfaces of the cross-section.

Figure 4 presents the newly proposed shear function compared to the parabolic shear deformation beam theories (PSDBT) suggested by Reddy (1984). It can be seen that the proposed shear functions in good agreement and similar with Reddy's analytical function curve. Higher-order Shear Deformation Theory which does not required to the shear correction factor, and using the shear function satisfies the stress-free boundary conditions, at both top and bottom fibers (z = ±h/2) of the FGM beams.

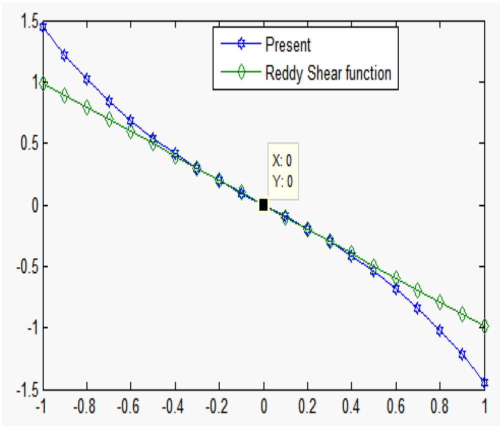

Fig.4. Comparison of analytical function curves. Please do not replace the current figure with another version having a white background; the original image should be retained.

5.2 Comparison of displacements

In the present investigation, the mechanical and geometrical properties of the C-S beam (Figure 3) are namely: Length L = 10m; Height h = 0.1m; Width b = 0.01m; Load q = P = 1Pa n which the Poisson's ratio remains constant $\upsilon=\mu=0.3$; E_c = 200GPa; E_m = 70GPa. According to Equations (19) and (20) consecutively, the new contribution proposed by the models inspired from research works of Ghugal and Sharma (2011) are not mentioned elsewhere. The maximum transverse displacement at the mid-span (x=L/2) for bending of thick C-S beams supporting concentrated load at the middle (w_{c3}) and another uniformly distributed load over the long span (w_{c4}), respectively, are given by the following expressions:

$$w_{C3}\left(x=\frac{L}{2}\right)=-\frac{5PL^3}{192E_xI}\left[\frac{1}{2}+1.92(1+\mu)\frac{h^2}{L^2}\right] \quad (19)$$

$$w_{C4}\left(x=\frac{L}{2}\right)=-\frac{3qL^4}{384E_xI}\left[1+5.76(1+\mu)\frac{h^2}{L^2}\right] \quad (20)$$

It is interesting to note that sandwich theorem is applicable for the case of the shear coefficient of C-S beam with mixed boundary conditions, because its maximum displacement value occurs between the maximum displacement values of C-C beam and S-S beam supported ends, respectively. Furthermore, the same remarks could be cited for the constitutive stress-strain relations because to be derivatives, which is reasonable. As proportional relation, it could be noted also that the rigidity value of the beam increases, the displacement decreases, and the inverse wise is correct. The increasing of the power law index leads to an increase in the displacement. A difference reached a rate equal to 0.71 ‰, which implies the correctness and the accuracy of the developed model (Figure5).

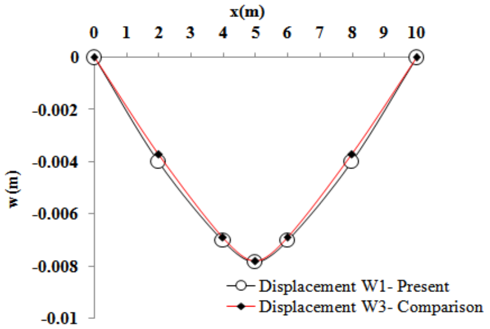

Fig.5. Comparison of displacements of C-S beam subjected to concentrated load predicted from the proposed model and those obtained from Ghugal model.

Figure 6 shows the typical distribution comparison of the transverse displacement through the span of the C-S beam under uniform load q.

It can be seen that the transverse displacements predicted from the proposed model are in good agreement with those obtained from Ghugal et al model. The difference between the max displacement values obtained from the both models is around 3‰ which confirms the similar concordance.

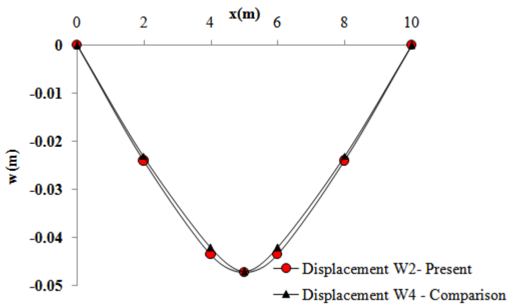

Fig.6. Comparison of displacements of C-S beam subjected to uniformly distributed load predicted from the proposed model and those obtained from Ghugal model.

Adding another example using steel beam (IPE 400) having the following properties: Type of steel beam: S275; Moment of inertia I = 23130 cm^4; Poisson's ratio μ = 0.3; Elasticity modulus E_x = 210GPa; Length L = 10 m; Distributed load q = 7 KN/m (including the self-weight of the beam and the pedestrian load). The displacement value at mid-span (x = L/2) evaluated from the proposed model (Eq.13-a) (w_{c2} = 0.11335 mm) is in good agreement with that predicted from Ghugal model (Eq.20) (w_{c4} = 0.11286 mm) with a displacement difference rate of 4.3 ‰ which proves the proposed model validity.Substituting the distributed load q by the concentrated load P=36kN applied at the mid-spanof the same previous steel beam (IPE 400), the displacement values obtained at x = L/2 from the proposed and Ghugalmodels are equal to w_{c1} = 0.0968944 mm (Eq.12-a) and to w_{c3} = 0.0966603mm (Eq.19), respectively, with a difference rate equal to 2.416 ‰ which confirms moreover the efficiency and accuracy of the developed model.

5.3 Shear stress

In Figures 7 and 8, the comparison shows a nonlinear parabolic curve distribution of transverse shear stresses, through the beam thickness, predicted from the proposed (present) model and the higher-order theories model. It can be observed that both curves exhibit a good convergence. For the first derived of polynomial shear function at both top and bottom fibers (z = ±h/2), the nullity of the shear stress at theses fibers can be confirmed. While, the shear stress is maximal at the midline fiber when φ(z = 0) =0. Furthermore, overestimate value of the shear stress meaning that the section can't support loadings and failure. Otherwise, underestimate value of stresses meaning that the section doesn't reach its critical state. At the maximum shear stress around 0.15x10^5 Pa, it can be seen that the shear difference between both models is about 3% only.

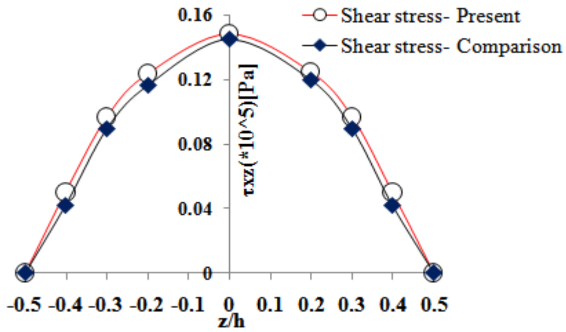

Fig.7. Shear stress (τ_{xz}) through thickness of C-S beam under concentrated load P predicted from proposed (Present) model and Ghugal model.

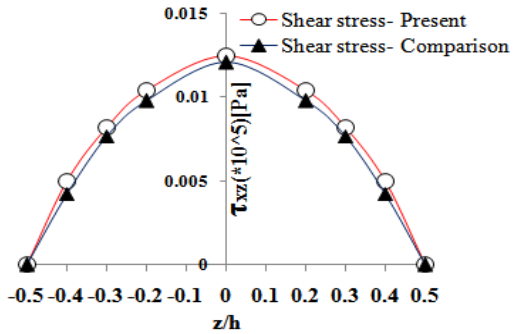

Fig.8. Shear stress(τxz) through thickness of C-S beam under uniformly distributed load q predicted from proposed (Present) model and Ghugal model.

It is noted that the shear stress controlling the design should ensure the splitting failure mode for safety, because researchers are mainly based on their contribution to the development of guidelines and standards. From where, significant damages frequently appear affecting the outside of FGM beams, if not respected recent rules developed on bending and free vibration and also buckling, in practice applications. It is also noted that the investigation of the numerical results of the comparative study allowed highlighting and has shown the possibility of their adequate performance for the composite structures to the design limits specified in the guidelines.

6 Conclusion

Clamped-simply supported (C-S) FGM beams with mixed boundary conditions is studied with efficacy deterministic manner as main objective, taking into account the shear strains in the displacement field, when higher-order shear strain new theories are used including a new developed polynomial shear function. From the comparison of the analyzed results, in terms of shear stress and displacement, obtained from the proposed model and existing models. The new mathematical model developed for C-S FGM beams taking into account the shear strains in the displacement field, shows the best similarity and convergence with that published in literature, as consequently, it is an efficient solution for C-S FGM beams under bending. The shear deformation utilizing the new polynomial shear function, implying stress-free at both limits bottom and top faces of the section through the total thickness, shows a good convergence with those of literature. The constitutive stress-strain relations exhibit a good agreement with those found in literature. The increasing of the power law index is limited by the real state value given by the experimental slope of materials. Although, the increasing of the power law index leads to an increase in the displacement. As concluding, the newly shear coefficient S evaluated is held as constant taking value of 3*S for C-SFGM beams with limits boundary condition. Finally, the FGM beams analysis requires establishing tools for modeling, including the calculation methods, which are essential for the design and verification of their safety in practice applications. The shear function proposed in this study can be used as a reference in the future researches. For future, it is recommended and suggested to investigate FGM structures based on machine learning with reliability and cost optimizations for other materials of beams and shells not mentioned in the present work, using the new proposed model. The presented theory could be considered as a reference for future research works. Furthermore, behind the curtain of the unknown, science continues to uncover other secrets in different ways and make significant contributions that advance the science of composite materials to broader horizons, and research continues to provide the best.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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List of symbols

b: Width; h: Thickness; L: Length.

D^*_{11}, F^*_{55}: Are bendingand shear stiffness constants of the beam, respectively.

M_m, M_c: Are material properties of metal and ceramic, respectively.

E_x, Gxz: Stiffness coefficients are Young modulus, and shear modulus, respectively.

υ=μ: Is the Poisson's ratio of the beam material.

P,q: Concentrated load, uniformly distributed load, respectively.

P: It is the power law index.
S: The shear coefficient.
HSDT: Hyperbolic shear deformation theory.
$\delta\varepsilon_x$, $\delta\gamma_{xz}$: are the incremental normal strain, incremental shear strain, respectively.
$\theta = \partial u/\partial z$; $\partial w_0/\partial x$: Are the rotation angles for axial and transverse displacements derivative, for z- and x-axes, respectively.
u, v, w: The displacement components (axial sens-x, axial sens-y, transverse sens-z), respective.
w_{c1}: The present novel transverse displacement of C-S beam subjected to concentrated load P at the middle.
w_{c2}: The present novel transverse displacement of C-S beam under uniformly distributed load q over the long span
w_{c3}: Ghugal and Sharma (2011)'s the maximum transverse displacement at the mid-span of C-S beams supporting concentrated load.
w_{c4}: Ghugal and Sharma (2011)'s the maximum transverse displacement at the mid-span of C-S beams supporting uniformly distributed load over the long span.
σ_x,τ_{xz}: Normal Stress, and shear stress, respectively.
ε_x, γ_{xz}: Normal Strain, and shear deformation, respectively.
φ (z); φ (z)ψ(x, z): Shear function, and warping shear, respectively.

Appendix-1

Materials and cross-section properties

$$E_{eff} = (E_c - E_m)(\frac{z}{h} + \frac{1}{2})^P + E_m$$

$$G_{eff} = (G_c - G_m)(\frac{z}{h} + \frac{1}{2})^P + G_m$$

Beam section characteristics expressions for extension, bending-extension couplings, and bending stiffness coefficients, are presented by the following equations:

$$D_{aa} = b\int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)dz = bh\left(E_m + \frac{E_c - E_m}{p+1}\right)$$

$$D_{ab} = b\int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)zdz = \frac{bh^2}{2}(E_c - E_m)[\frac{p}{(p+1)(p+2)}]$$

$$D_{bb} = b\int_{\frac{-h}{2}}^{\frac{h}{2}} E(z)z^2dz$$

$$= \frac{bh^2}{12}[3(E_c - E_m)\frac{p^2+p+2}{(p+3)(p+2)(p+1)}]$$

The coefficient transverse of shear stiffness is presented as follows:

$$D_s = k_z b\int_{\frac{-h}{2}}^{\frac{h}{2}} G(z)dz$$

Where, (k_z) is defined as the shear correction factor for bending.

$$b\int_{x=0}^{x=L}(N\frac{\partial\delta u_0}{\partial x} - M_b\frac{\partial^2\delta w_0}{\partial x^2} + M_s\frac{\partial\delta\psi_X}{\partial x} + Q\delta\psi_X)dx - b\int_{x=0}^{x=L} q\delta w_0 dx = 0$$

The following equations present the normal resultant force (N) and the shear resultant force (Q), respectively.

$$N = \int_{\frac{-h}{2}}^{\frac{h}{2}}\sigma_x dz \;\; ; \;\; Q = \int_{\frac{-h}{2}}^{\frac{h}{2}}\frac{\partial\varphi(z)}{\partial z}\tau_{zx} dz$$

In addition, the following equations present the flexure moment (M_b) and the shear flexure moment (M_s), respectively:

$$M_b = \int_{\frac{-h}{2}}^{\frac{h}{2}} z\sigma_x dz \qquad ;M_s = \int_{\frac{-h}{2}}^{\frac{h}{2}}\varphi(z)\sigma_x dz$$

It can be noted that φ(z) is the shape function. From the above integrals, the equilibrium equations are derived as follows:

$$\frac{\partial N(x)}{\partial x} = 0; \; \frac{\partial^2 M_b}{\partial x^2} + q = 0;$$

$$\frac{\partial^2 M_S}{\partial x^2} + \frac{\partial Q}{\partial x} + q = 0; \;\; \frac{\partial M(x)}{\partial x} - N(x) = 0$$

Appendix-2

The basis of a simplified approach was introduced in detailssteps todemonstrate the shear coefficient S of C-S beams, necessary to checking two parts:

First part:

Beamwith loading at the mid-span of the deformed which imposesto using the intrinsic symmetric hypothesis condition as seen:

$$Atx = \frac{L}{2} => \varphi_x\left(\frac{L}{2}\right) = 0$$

This condition tolerated and could be separated the restraints of beam as well as Clamped-Clamped and Simply- Simply. It is based on the expression of the bending moment M(x) given as follows:

$$\frac{\partial M_0}{\partial x} = bhG_{xz}\left(\varphi_x + \frac{\partial w_0}{\partial x}\right)$$

After substitution and simplification, the final version of the shear coefficient S is defined as follows: $S_{C-C} = 5\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2$ for (Clamped-Clamped) beam; and also $S_{S-S} = \frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2$ for (Simply- Simply) beam.

Second part:

For gathering solution needed to application of ***Sandwich* Theorem** (at x=L/2) to annulated the intrinsic symmetric hypothesis, the final version of solution can be obtained as follows:

$\frac{1}{2}[S_{C-C} + S_{S-S}] = 3\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2$ for C-S beam.

It is clearly that already apparent as fewer details extracting was able for helping to achieve hide expression of displacements. The same meaning for the beam expression of displacement as shown below:

C-S Beam with central concentrated load

First derivation of displacement for (Simply-Simply) beam and too (Clamped-Clamped) beam, with central concentrated load P, are given, respectively, by the following expressions:

$$\frac{w_{S-S}(x)}{\partial x} = \frac{P}{4E_x bh^3}(-12x^2 + 3L^2) \; ;$$

$$\frac{w_{C-C}(x)}{\partial x} = \frac{P}{4E_x bh^3}(-12x^2 + 6Lx)$$

After integration and simplification and gathering solution, needed to application of **Sandwich Theorem** (at x = L/2) to annulated the intrinsic symmetric hypothesis, the final version of solution can be obtained as follows:

$$w_1(x) = \frac{1}{2}[w_{S-S}(x) + w_{C-C}(x)]$$

The final expression is presented by Eq. (12). The maximum value is calculated at (x=L/2) by Eq. (12-a).

C-S Beam with uniformly distributed load

First derivation of displacement W_{S-S}(x) for (Simply-Simply) beam, and W_{c-c}(x) for (Clamped-Clamped) beam with uniformly distributed load q, are given by the following expressions, respectively:

$$\frac{w_{S-S}(x)}{\partial x} = \frac{q}{2E_x bh^3}(4x^3 - 6Lx^2 + L^3)\ ;$$

$$\frac{w_{C-C}(x)}{\partial x} = \frac{q}{2E_x bh^3}(4x^3 - 6Lx^2 + 2L^2 x)$$

After integration and simplification and gathering solution, needed to application of **Sandwich Theorem** (at x = L/2) to annulated the intrinsic symmetric hypothesis condition, it can be obtained the target displacement. The final version of solution is given as follows:

$$w_2(x) = \frac{1}{2}[w_{S-S}(x) + w_{C-C}(x)]$$

The general displacement expression of C-S beam under concentrated load applied at mid-span:

$$w_1(\mathrm{x}) = -\frac{\mathrm{P}L^3}{4\,E_x \mathrm{b}h^3}\left(-4\frac{x^3}{L^3} + \frac{3}{2}\frac{x^2}{L^2} + \frac{3}{2}\frac{x}{L} + 3\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2\right)$$

The general displacement expression of C-S beam under uniformly distributed load:

$$w_2(\mathrm{x}) = -\frac{\mathrm{q}L^4}{4\,E_x \mathrm{b}h^3}(2\frac{x^4}{L^4} - 4\frac{x^3}{L^3} + \frac{x^2}{L^2} + \frac{x}{L} + 3\frac{E_x}{G_{xz}}\left(\frac{h}{L}\right)^2)$$