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Analytical solutions of 1D population balance equation at steady-state

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ABSTRACT

Due to their effectiveness in separation and purification, two-phase flow columns (liquid-liquid, gas-liquid, and solid-liquid) are extensively utilized in the chemical industries. PBE has recently been recognized as an appropriate tool for modeling this kind of column owing to its ability to describe both the hydrodynamics and the mass transfer of the dispersed phase. In this work, we solved analytically one-dimensional PBE at steady-state using the Adomian decomposition method and the Method of moments. Analytical solutions are provided for pure growth, pure breakup, breakup with growth, pure aggregation, and breakup with growth with aggregation. The obtained results encourage extending the applicability of both methods to solve 1D PBE.

KEYWORDS

Population balance
Multiphase column
Adomian decomposition method
Method of moments

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1 Introduction

Mass and heat transfer improvement is one of the most intriguing features of dispersed phase systems. This system consists of two phases, continuous and dispersed, the latter is a set of particles (crystals, droplets, bubbles, cells, ...), its properties are subjected to change due to several phenomena: breakup, aggregation and growth. The population balance equation describes these processes and particle movement in the physical space. Since the PBE was used efficiently for modeling many processes of crystallization, absorption, extraction, granulation and polymerization, it became indispensable to modelling the dispersed phase systems [1].

The population balance equation is classified as a partial integro-differential equation. It shows difficulties in finding its exact solution. However, the researchers found some exact solutions just for a few simple cases, and comprehensive propositions and development of numerical methods have been made to find approximated solutions [2]. The well-known numerical methods can be regrouped under these three families: methods of moments, stochastic methods and class methods. Articles are concerned with methods for solving the PBE [3-6].

In recent years, semi-analytical methods find diverse applications in engineering because of their advantages: simplicity in programming and eliminating those complications of the calculations produced by the discretization in the numerical methods [7, 8]. Math software development also has directly contributed to using these methods extensively. Adomian Decomposition Method, Perturbation Homotopic Method, Variational Iteration Method and successive generation method, four semi-analytical methods have been applied to solve the population balance equations for a batch reactor, continuous reactor and columns [9-13].

The analytical solutions are undoubtedly needed to test the accuracy of the numerical methods. The population balance equation without convective term has a limited set of exact solutions [14-22]. Its solutions in moments and particle number distributions are routinely compared with the numerical solutions in hundreds of scientific papers. One-dimensional PBE is rarely has analytical solutions. However, in [9], analytical and semi-analytical solutions were provided by assuming that the particle velocity is uniform. Analytical solutions are given using the successive generation method for pure convection and convection with particle absorption. Semi-analytical solutions are developed using the method of characteristics for a more realistic case, that is, convection with particle absorption and breakage. In [3] they also assumed uniform particle velocity and applied the chain rule with Laplace transform to find the exact solutions for convection with breakup and convection with aggregation. They also developed an integrated solution for pure convection, which can model different forms of feed distribution and particle velocity. The last idea was extended by applying the variational iteration method to find other analytical solutions for a constant particle velocity for different processes: pure growth, pure breakage, pure coalescence, breakage with growth, coalescence with growth and breakage with coalescence [23]. They also solved the PBE analytically for a volume-dependent particle velocity for pure breakage and pure coalescence [23] [24], they introduced an analytical methodology based on the Laplace transformation and the Adomian Decomposition Method to solve the one-dimensional population balance equation.

This study aims to provide analytical solutions of nondynamic PBE with the presence of the advection in the physical space and particle–particle interactions. Adomian Decomposition Method and method of moments are applied separately for simple and combined problems.

2 Steady-state 1-D population balance equation

Considering the presence of breakup, aggregation, growth, and transport events, the population balance equation can describe the steady-state behavior of the dispersed phase along the column height z [1]:

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$$\frac{\partial}{\partial z}\big(v_d(z,v)n(z,v)\big) = \frac{\partial}{\partial z}D_d(z)\frac{\partial}{\partial z}n(z,v) + \frac{Q_d}{A}n_{in}(v)\delta(z-z_d) + H_a(z,v) + H_b(z,v) + H_g(z,v) \quad (1)$$

The left-hand side of the above equation is well-known as a convection-diffusion term. The particle movements inside the column are characterized by particle velocity v_d and dispersion coefficient D_d. In the following, we will define all terms in the right hand of the equation (1):

The dispersed phase feeds the column with an inflow distribution n_{in} and a flow rate Q_d at the level z_d, this source point is modeled by [25]:

$$\frac{Q_d}{A}n_{in}(v)\delta(z-z_d) \quad (2)$$

where: $\delta()$, A :is the Dirac-delta function and the cross-sectional column area, respectively.

2.1 Breakup term

The breakup process is expressed as follows:

$$H_b(z,v) = \int_v^\infty \beta(u,v)g(z,u)n(z,u)du - g(z,v)n(z,v) \quad (3)$$

It consists of two nonlinear terms birth and death. The birth term is given by an integral from v to ∞, which considers the breakup of the particles has a volume $u, u \geq v$. The death term is a negative function; it is a production of the daughter particles from the mother particle that has a volume v. $g(z,v)$ is the breakup frequency and $\beta(u,v)$ is the particle daughter particle distribution.

2.2 Aggregation term

Modeling the aggregation is far more complicated than modeling the breakup. However, it can be written as:

$$H_a(z,v) = \frac{1}{2}\int_0^v \omega(z,v-u,u)n(z,v-u)n(z,u)du - n(z,v)\int_v^\infty \omega(z,v,u)n(z,u)du \quad (4)$$

These two terms on the right-hand side of this equation represent the generation and the loss attributed to the aggregation, respectively. Where $\omega(z,v,u)$ is the aggregation rate.

2.3 Growth term

The growth phenomenon is described by this derivative:

$$H_g(z,v) = -\frac{\partial(G(z,v)n(z,v))}{\partial v} \quad (5)$$

where, $G(z,v)$ is the growth rate. We can rewrite the equation (1) as [3]:

$$\frac{\partial}{\partial x}(v_d(x,v)n(x,v)) = \frac{\partial}{\partial x}D_d(x)\frac{\partial}{\partial x}n(x,v) + H_a(x,v) + H_b(x,v) + H_g(x,v) \quad (6)$$

where $x = z - z_d$ and the boundary condition is given by:

$$n(0,v) = \frac{Q_d}{A}n_{in} \quad (7)$$

3 Adomian decomposition method

Adomian decomposition was developed by George Adomian, it is a widely used analytical approach for solving algebraic, differential, integral, and integro-differential equations by formulating the solution as an infinite power series that converges to the exact solution [26, 27]. This method has various applications to solve the population balance equation [11, 12].

4 Application

In this application, we proposed five problems: pure growth, pure breakup, pure aggregation, simultaneous breakup with growth and a combination between three processes growth, breakup and aggregation. We applied the Adomian decomposition method for a pure breakup, pure aggregation and simultaneous breakup with growth. For the last problem, we used the method of the moments due to its complexity in solving. The diffusion term is neglected, and all the variables are dimensionless in all these problems. Except for pure growth, we tried to consider the agitation effects of the rotor on the dispersed phase. Since the system is non-homogeneous, we can consider the rate of any process as a function of the height z. So, we choose an exponential function; it is, kz^a, where k and a are constants and ≥ 0. We consider a two-phase column with the following dimensions: $A = 1, z_d = 0.25, h = 2.7$. Applications in details are provided in the following sections:

4.1 Pure growth

In this specific problem, we assume that the particles only undergo growth and that the growth rate depends on the particle's volume $G(z,v) = k_g v$, where k_g is a constant and $k_g \geq 0$. As a result of the mass transfer, the dispersed phase grows whenever it absorbs components from the continuous phase. The particles rise with a linear particle velocity that varies with particle size:

$$v_d(z,v) = (k_d + k_v d) \quad (8)$$

where d is the particle diameter $d = \sqrt[3]{v/c_v}$, c_v is the form factor, the particle might be spherical $c_v = \pi/6$ or cubic $c_v = 1$. The particle velocity is also adjusted by two parameters which must investigate the following conditions:

$$k_d > 0, k_v > 0, \text{constants} \quad (9)$$

The proposed formula of the particle velocity can overcome the retention of the small particles in the column.

For only growth, PBE should be written as:

$$\frac{\partial}{\partial x}(v_d(x,v)n(x,d)) = -\frac{\partial(G(x,v)n(x,v))}{\partial v} \quad (10)$$

This equation is solved analytically, the exact solution in both internal and external coordinates for any feed distribution n_{in}, is given by:

$$n(x,v) = \frac{Q_d k(x,v)}{Av}n_{in}(k(x,v)) \quad (11)$$

The function $k(z,v)$ has the following formula:

$$k(x,v) = c_v\left(\frac{k_d}{k_v}\right)^3 \text{ProductLog}\left[\left(\frac{v}{c_v}\left(\frac{k_v}{k_d}\right)^3 e^{\frac{3k_v v^{1/3}c_v^{-1/3} - k_g x}{k_d}}\right)^{1/3}\right]^3 \quad (12)$$

where: ProductLog is the product logarithm function, also called the Lambert W function.

If we replace x by $z - z_d$ in the above solution we get:

$$n(z,v) = \begin{cases} 0, z < z_d \\ \frac{Q_d k(z-z_d,v)}{Av}n_{in}(k(z-z_d,v)), \text{otherwise} \end{cases} \quad (13)$$

For $k_g = 0$, the above solution provides a solution for pure convection at steady state. The same solution can be derived from the dynamic convection solution of [9].

In Figure 1 we investigated the impact of the parameter k_v on the growth process. It represents analytical distributions at the column output for various experiments: the first experiment $k_v = 1$, the second experiment $k_v = 2$, the third experiment $k_v = 3$. They were done with $Q_d = 0.15, A = 1, c_v = \pi/6, k_d = 0.15$.

The inflow distribution for this case is taken as follows:

$$n_{in} = e^{-v} \tag{14}$$

The Figure 1 is shown that the distribution shape is distinct for each test, and the effect of the velocity appears very clear. These tests demonstrate that the mean volume of particles increases, and the total number of the leaving particles is reduced much with the largest value of k_v, the mass transfer takes place more efficiently when the particles move slowly.

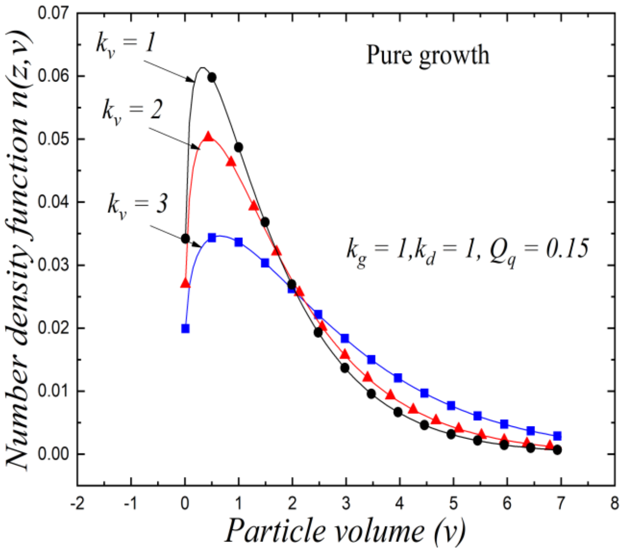

Fig.1. Effect of the velocity coefficient k_v on the particle number density function at the column outlet.

4.2 Pure aggregation

The second problem is that under these conditions, the particles only aggregate at a rate proportional to the volume of the particles and the height of the column, $\omega(v,u) = k_c z^a vu$. The upward movement of particles is linear and proportional to their volume, $v_d = k_v v$, where $k_v > 0$. The inflow distribution is exponential, $n_{in} = e^{-v}$.

This problem was introduced first by [23], where they considered $k_c = 1, a = 0$ and $k_v = 1$. For convection with aggregation, equation (6) is reduced to:

$$\frac{\partial}{\partial x}(v_d(x,v)n(x,d)) = \frac{1}{2}\int_0^v \omega(x,v-u,u)n(x,v-u)n(x,u)du - n(x,v)\int_v^\infty \omega(x,v,u)n(x,u)du \tag{15}$$

By applying ADM to the above equation, that leads:

$$n_{i+1}(x,v) = \frac{1}{v_d(v)}\int_0^x \frac{1}{2}\int_0^v \omega(x,v-u,u)n_i(x,v-u)n_i(x,u)dudx - \frac{1}{v_d(v)}\int_0^x n_i(x,v)\int_v^\infty \omega(x,v,u)n_i(x,u)dudx \tag{16}$$

We have the following component solutions:

$$n_0 = \frac{Q_d}{A}e^{-v} \tag{17}$$

$$n_1 = \frac{e^{-v}k_c Q_d^2(-12+v^2)x^{1+a}}{12(1+a)A^2 k_v} \tag{18}$$

$$n_2 = \frac{e^{-v}k_c^2 Q_d^3(360-60v^2+v^4)x^{2+2a}}{480(1+a)^2 A^3 k_v^2} \tag{19}$$

These can be simplified as:

$$n_i = \sum_{i=1}^{\infty} \frac{Q_d^n}{A}\frac{4(1+a)^2 k_c^{i-1}k_v^2 x^{(i-1)(1+a)}v^{2(-1+i)}e^{-v}}{\text{Gamma}\,[2i]\left(2(1+a)k_v+k_C\frac{Q_d}{A}x^{1+a}\right)^{i+1}} \tag{20}$$

The closed-form solution is:

$$n(x,v) = \frac{4(a+1)^2(\sqrt{x})^{-a-1}e^{-v}\sqrt{\frac{Q_d}{A}}k_v^2}{\sqrt{k_c}v\left(2(a+1)k_v+k_c\frac{Q_d}{A}x^{a+1}\right)^{3/2}}\chi \tag{21}$$

where

$$\chi = \text{Sinh}\left[\sqrt{\frac{Q_d}{A}}k_c v(\sqrt{x})^{a+1}/\sqrt{2(a+1)k_v+k_c\frac{Q_d}{A}x^{a+1}}\right] \tag{22}$$

By replacing x by $z - z_d$ in the above solution, we get:

$$n(z,v) = \begin{cases} 0, & z < z_d \\ \frac{Q_d}{A}e^{-v}, & z = z_d \\ \frac{4(a+1)^2\left(\sqrt{z-z_d}\right)^{-a-1}e^{-v}\sqrt{\frac{Q_d}{A}}k_v^2}{\sqrt{k_c}v\left(2(a+1)k_v+k_c\frac{Q_d}{A}x^{a+1}\right)^{3/2}}\chi, & z > z_d \end{cases} \tag{23}$$

where:

$$\chi = \sinh\left[\sqrt{\frac{Q_d}{A}}k_c v\left(\sqrt{z-z_d}\right)^{a+1}/\sqrt{2(a+1)k_v+k_c\frac{Q_d}{A}(z-z_d)^{a+1}}\right] \tag{24}$$

The obtained solution was introduced by [23] for dynamic 1-D PBE.

We examine the holdup and outflow distribution profiles for different values of the exponent a; considering three test cases: $a = 0, 1.5$ and 3, analytical and numerical results are shown in Figure 2.

Figure 2(b) can be seen that the holdup profile is linear for $a = 0$, but it curves out for $a > 0$, for $a > 0$, these nonlinear rates become stronger with an increase in base z; their curves can be split into two sections: a weak aggregation section ($z \in [z_d, 1.5]$) and a strong aggregation section ($z \in [1.5, 2.7]$).

Figure 2(a) it chows that a critical drop in the distribution at the input of the column by dropping the a value.

Large-volume particles are produced during the aggregation process when the particle velocity depends on the particle volume, giving each particle a short residence time. In other words, that decreases the existence of the population in the column. When the value of the parameter a is increased, there is a corresponding drop in the outflow moments.

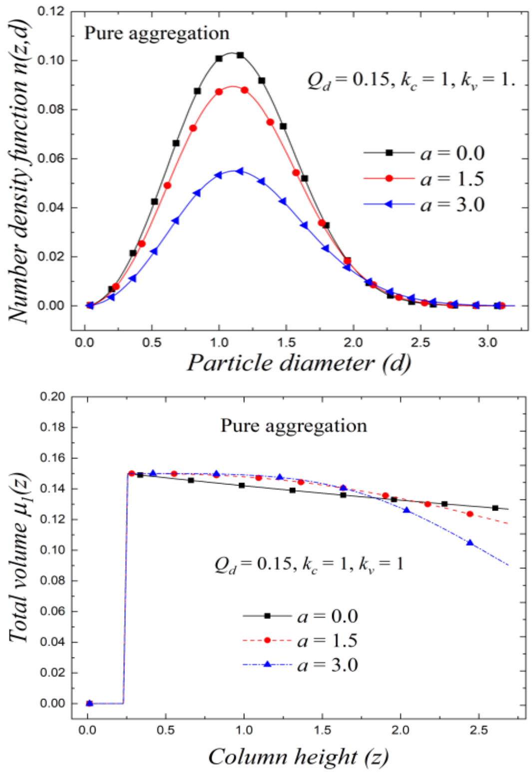

Fig.2. Effect of the aggregation parameter a for pure aggregation.

4.3 Pure breakup

During the presence of the particle in the column, the breakup process is active with $g(z,v) = k_b z^a v, \beta(u,v) = \frac{2}{u}$, and a constant particle velocity, $v_d > 0$. We consider two particular cases:

Case 01, the feed distribution the first-order gamma distribution:

$$n_{in} = ve^{-v} \tag{25}$$

Case 02, the feed distribution is a normalized Gaussian distribution with the mean m and the stadaed diviation s :

$$n_{in} = \frac{e^{-\frac{(v-m)^2}{2s^2}}}{B}, B = \sqrt{2\pi}s \tag{26}$$

If the operations inside the column allow the particles undergo only breakup, the model looks like:

$$\frac{\partial}{\partial x}(v_d(x,v)n(x,v)) = \int_v^{\infty} \beta(u,v)g(x,u)n(x,u)du - g(x,v)n(x,v) \tag{27}$$

By applying ADM to the above equation, that leads:

$$n_{i+1}(x,v) = \frac{1}{v_d}\int_0^x \int_v^{\infty} \beta(u,v)g(x,u)n_i(x,u)du - g(x,v)n_i(x,v)dx \tag{28}$$

For the first case, we have the following component solutions:

$$n_0 = \frac{Q_d}{A} ve^{-v} \tag{29}$$

$$n_1 = -\frac{e^{-v}k_b Q_d(-2-2v+v^2)x^{1+a}}{(1+a)Ak_v} \tag{30}$$

$$n_2 = \frac{e^{-v}k_b^2 Q_d(4-2v-4v^2+v^3)x^{2+2a}}{2(1+a)^2 A k_v^2} \tag{31}$$

These can be simplified as:

$$n_i = \frac{e^{-v}Q_d}{A}\left(v + \sum_{i=1}^{\infty} \frac{v^{i-2}(2(-1+i)i+(-3i+i^2)v-2iv^2+v^3)}{i!(-1)^i(1+a)^i k_v^i k_b^{-i} x^{-i-ia}}\right) \tag{32}$$

The closed-form solution is:

$$n(x,v) = \frac{Q_d(k_v+ak_v+k_b x^{1+a})((1+a)k_v v+k_b(2+v)x^{1+a})}{e^{v\left(1+\frac{k_b x^{1+a}}{k_v+ak_v}\right)}(1+a)^2A^2k_v^2} \tag{33}$$

By replacing x by $z - z_d$ in the above solution, we get:

$$n(z,v) = \begin{cases} 0, z < z_d \\ \frac{Q_d(k_v+ak_v+k_b(z-z_d)^{1+a})((1+a)k_v v+k_b(2+v)(z-z_d)^{1+a})}{e^{v\left(1+\frac{k_b(-z_d)^{1+a}}{k_v+ak_v}\right)}(1+a)^2A^2k_v^2}, \text{otherwise} \end{cases} \tag{34}$$

For the second case, ADM gives:

$$n_0 = \frac{Q_d}{A}\frac{e^{-\frac{(v-m)^2}{2s^2}}}{B} \tag{35}$$

$$n_1 = \frac{Q_d}{A}\frac{k_b x^{(1+a)}}{B(1+a)k_v}\left(-ve^{-\frac{(m-v)^2}{2s^2}} + \sqrt{2\pi}s\left(1+\text{Erf}\left[\frac{m-v}{\sqrt{2}s}\right]\right)\right) \tag{16}$$

$$n_2 = \frac{Q_d}{A}\frac{k_b^2 x^{2(1+a)}}{2B(1+a)^2k_v^2}\left((2s^2+v^2)e^{-\frac{(m-v)^2}{2s^2}} + \cdots\right) \tag{37}$$

These can be simplified, to be:

$$n_i = \frac{Q_d}{BA}\frac{e^{-\frac{(m-v)^2}{2s^2}}(k_v-k_b v x)+k_b\sqrt{2\pi}sx\left(1+\text{Erf}\left[\frac{m-v}{\sqrt{2}s}\right]\right)}{(1+a)k_v} + \frac{Q_d}{BA}\sum_{i=2}^{\infty}\frac{(-1)^i k_b^i k v^{-i} v^{-2+i} x^i}{2\text{Gamma}[-1+i](1+a)^i}\chi_i \tag{38}$$

where

$$\chi_i = 2e^{-\frac{(m-v)^2}{2s^2}}\left(s^2+\frac{v^2}{(-1+i)i}\right)+\sqrt{2\pi}s\left(m-\frac{(1+i)}{-1+i}v\right)\left(1+\text{Erf}\left[\frac{m-v}{\sqrt{2}}\right]\right) \tag{39}$$

The exact solution is:

$$n(x,v) = \frac{Q_d}{2(1+a)^2ABk_v^2}e^{-\frac{(m-v)^2}{2s^2}-\frac{k_b v x^{1+a}}{k_v(1+a)}}\chi(x,v) \tag{40}$$

where

$$\chi(x,v) = 2(1+a)^2 k_v^2 + 2(1+a) e^{\frac{(m-v)^2}{2s^2}} k_b k_v \sqrt{2\pi} s x^{1+a} + k_b^2 s \left(2s + e^{\frac{(m-v)^2}{2s^2}} \sqrt{2\pi}(m-v)\right) x^{2+2a}$$
$$+ \frac{k_b \sqrt{2\pi} s}{e^{-\frac{(m-v)^2}{2s^2}}} x^{1+a} (2(1+a) k_v + k_b (m-v) x^{1+a}) \mathrm{Erf}\left[\frac{m-v}{\sqrt{2s}}\right] \tag{41}$$

By replacing x by $z - z_d$ in the above solution, we get:

$$n(z,v) = \begin{cases} 0, z < z_d \\ \frac{Q_d}{2(1+a)^2 A B k_v^2} e^{-\frac{(m-v)^2}{2s^2} - \frac{k_b v (z-z_d)^{1+a}}{k_v(1+a)}} \chi(z,v), \text{otherwise} \end{cases} \tag{42}$$

where

$$\chi(z,v) = 2(1+a)^2 k_v^2 + 2(1+a) e^{\frac{(m-v)^2}{2s^2}} k_b k_v \sqrt{2\pi} s (z-z_d)^{1+a} + k_b^2 s \left(2s + e^{\frac{(m-v)^2}{2s^2}} \sqrt{2\pi}(m-v)\right) (z-z_d)^{2+2a}$$
$$+ \frac{k_b \sqrt{2\pi} s}{e^{-\frac{(m-v)^2}{2s^2}}} \frac{(2(1+a) k_v + k_b (m-v)(z-z_d)^{1+a})}{(z-z_d)^{-1-a}} \mathrm{Erf}\left[\frac{m-v}{\sqrt{2s}}\right] \tag{43}$$

Table 1. Summary of the test case simultaneous growth and breakup.

$n_{in}(v)$	$g(z,v)$	$\Gamma(v,u)$	$G(z,v)$	$v_d(z,v)$
ve^{-v}	$k_b v z^a$	$2/u$	$k_g v z^a$ where $k_g = k_b$	$k_v, k_v > 0$

The analytical solution is usually derived from the feed distribution.

Figure 3 shows input and output distributions for case 01 and 02. We select the following set of parameters to achieve this graphical representation: $Q_d = 0.1, k_v = 1, k_b = 0.1, a = 0$. In the second case, the parameters m and s are $2, 0.7$, respectively. This figure demonstrates that when particles break up, the average particle volume decreases while the total number of particles reaches.

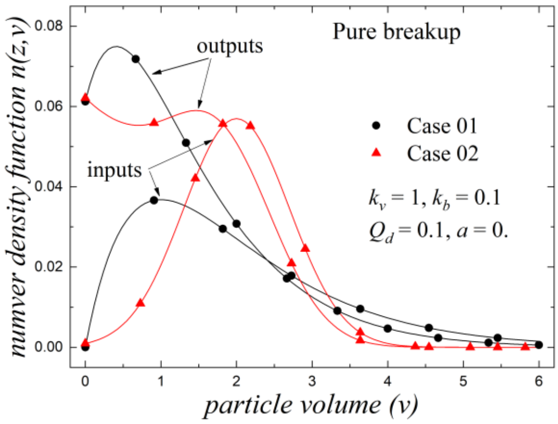

Fig.3. Comparison of input and output particle number density functions for two different feed distributions in the case of pure breakage.

4.4 Growth with breakup

We combined here growth with breakup processes. The table 1 presents the essential information, including breakup frequency daughter particle distribution growth rate particle velocity and feed distribution.

The governing equation for the considering problem is:

$$\frac{\partial}{\partial x}(v_d(x,v) n(x,v)) = \int_v^\infty \beta(u,v) g(x,u) n(x,u) du - g(x,v) n(x,v) - \frac{\partial (G(x,v) n(x,v))}{\partial v} \tag{44}$$

By ADM, we found this general form:

$$n_{i+1}(x,v) = \frac{1}{v_d} \int_0^x \int_v^\infty \beta(u,v) g(x,u) n_i(x,u) du - g(x,v) n_i(x,v) dx - \frac{1}{v_d} \int_0^x \frac{\partial (G(x,v) n_i(x,v))}{\partial v} dx \tag{45}$$

ADM gives:

$$n_0 = \frac{Q_d}{A} v e^{-v} \tag{46}$$
$$n_1 = \frac{2 k_b x^{1+a}}{k_v (1+a)} \frac{Q_d}{A} e^{-v} \tag{47}$$
$$n_2 = \frac{2 k_b^2 x^{2+2a}}{k_v^2 (1+a)^2} \frac{Q_d}{A} e^{-v} \tag{48}$$

These can be simplified, to be:

$$n_i = \frac{Q_d e^{-v}}{A} \left(v + \sum_{i=1}^{\infty} \frac{k_b^i x^{i+ia}}{\frac{1}{2} \mathrm{Pochhammer}[1,n] (1+a)^i k_v^i}\right) \tag{49}$$

The exact solution is:

$$n(x,v)=\frac{e^{-v}Q_d}{A}\left(-2+2e^{\frac{k_k x^{1+a}}{k_v(1+a)}}+v\right) \quad (50)$$

By replacing x by $z-z_d$ in the above solution, we have:

$$n(z,v)=\begin{cases}0, z<z_d\\ \frac{e^{-v}Q_d}{A}\left(-2+2e^{\frac{k_b(z-z_d)^{1+a}}{k_v(1+a)}}+v\right), \text{otherwise}\end{cases} \quad (51)$$

The entering mass of the dispersed phase depends on dispersed flow rate, inlet distribution and cross-sectional column area. The distribution function is presented in the Figure 4, We fixed all the parameters ($k_v=1, k_b=k_g=0.2, a=0$) and varied the value of the dispersed flow rate.

We can deduce from this figure the greatest Q_d value produces the greatest values of μ_1, and the dispersed flow rate influences the total mass of the dispersed phase along the column, that was presented experimentally for a liquid-liquid column [28].

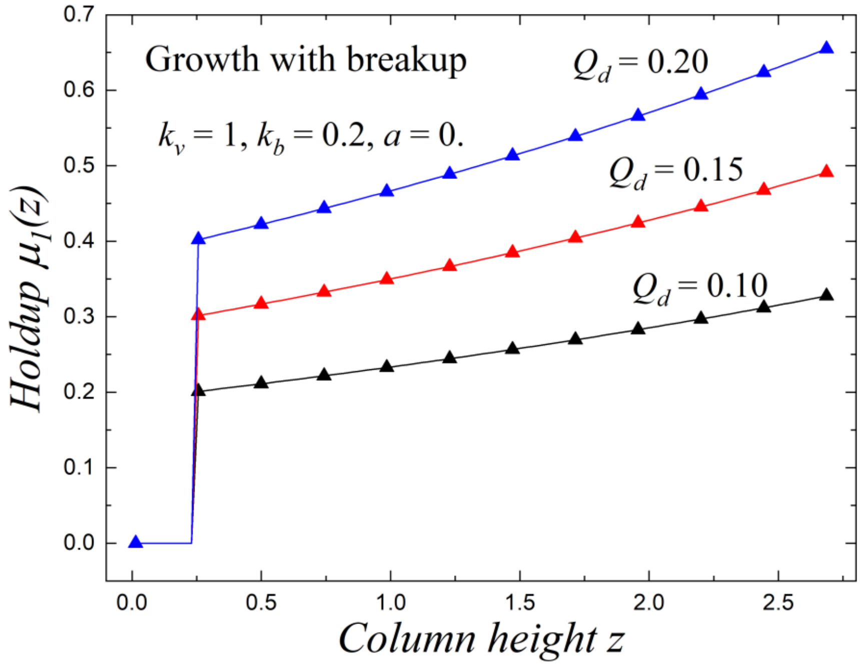

Fig.4. Effect of the dispersed phase flow rate on the holdup of the dispersed phase for the case of simultaneous growth and breakup.

4.5 Simultaneous growth and aggregation and breakup

A combination of three processes is made in this section. This problem creates more difficulties in getting its exact solution from the solution series that ADM computes. Alternatively, we applied the method of moments that allows getting the moments of the number density function directly without conserving the shape of the distribution. The details (breakup and aggregation frequencies, growth rate particle velocity and feed distribution) are summarized in table 2.

Table 2. Summary of the test case simultaneous growth and breakup.

$\omega(z,v,u)$	$g(z,v)$	$\Gamma(v,u)$	$G(z,v)$	$v_d(z,v)$
k_c	k_b	$2/u$	$k_g v$	$k_v, k_v\mu_1, k_v>0$

PBE, when without including the diffusive term, is expressed as:

$$\frac{\partial}{\partial x}(v_d(x,v)n(x,v))=H_a(x,v)+H_b(x,v)+H_g(x,v) \quad (52)$$

The moments transformation is given by [29]:

$$\mu_j(x)=\int_0^\infty v^j(x,v)dv \quad (53)$$

The moments of the particle size distribution provide compact quantitative measures of the dispersed phase evolution. The zeroth moment tracks the total number of particles, while the first moment corresponds to the total dispersed volume, allowing direct assessment of mass conservation or loss due to breakup and aggregation.
By applying MOM to each term of the above equation, we have [1, 21]:

$$\int_0^\infty v^j\frac{\partial}{\partial x}(v_d(x)n(x,v))dv=\frac{\partial}{\partial x}\left(k_v\mu_j(x)\right) \quad (54)$$

$$\int_0^\infty k_b n(x,v)v^j dv=k_b\mu_j(x) \quad (55)$$

$$\int_0^\infty v^j\int_v^\infty k_b\frac{2}{u}n(x,u)dudv=k_b\frac{2\mu_j(x)}{j+1} \quad (56)$$

$$\int_0^\infty v^j\frac{\partial(k_g v n(x,v))}{\partial v}dv=k_g j\mu_j(x) \quad (57)$$

$$\int_0^\infty v^j n(x,v)\int_v^\infty k_c n(x,u)dudv=k_c\mu_j(x)\mu_0(x) \quad (58)$$

$$\frac{1}{2}\int_0^\infty v^j\int_0^v k_c n(x,v-u)n(x,u)dudv=\frac{k_c}{2}\sum_{r=0}^{j}\binom{j}{r}\mu_r(x)\mu_{j-r}(x) \quad (59)$$

Finally, we collect all the above moment terms to have the following system of j ordinary differential equations:

$$\frac{\partial}{\partial x}\left(v_d(x)\mu_j(x)\right) = \left(1 - \frac{2}{j+1}\right)k_b\mu_j(x) + \frac{k_c}{2}\sum_{r=0}^{j}\binom{j}{r}\mu_r(x)\mu_{j-r}(x) - k_c\mu_j(x)\mu_0(x) - k_g j\mu_j(x) \tag{60}$$

More details about these mathematical simplifications and MOM applications are provided in [21]. We solved analytically the equation (59) by considering the feed distribution is that given by (25), and we proposed two particular cases of the particle velocity: constant $v_d(z) = k_v$ and nonlinear function $v_d(z) = k_v\mu_1(z)$

For $j = 0,1$ and 2, the exact solutions in the original variable z are listed as follows:

Case 01

In the nonactive zone, $z < z_d$:

$$\mu_j(z) = 0 \tag{61}$$

In the active zone, $z \geq z_d$:

$$\mu_0(z) = \frac{2e^{\frac{k_b(z-z_d)}{k_v}}k_bQ_d}{2Ak_b + \left(-1 + e^{\frac{k_b(z-z_d)}{k_v}}\right)k_cQ_d} \tag{62}$$

$$\mu_1(z) = \frac{Qd}{A}e^{\frac{k_g(z-z_d)}{k_v}} \tag{63}$$

$$\mu_2(z) = \frac{Q_d}{A^2k_b}e^{-\frac{(k_b-6k_g)(z-z_d)}{3k_v}}\left(2Ak_b + 3\left(-1 + e^{\frac{k_b(z-z_d}{3k_v}}\right)k_cQ_d\right) \tag{64}$$

Case 02

In the nonactive zone, $z < z_d$:

$$\mu_j(z) = 0 \tag{65}$$

In the active zone, $z \geq z_d$:

$$\mu_0(z) = \frac{2^{1-\frac{2k_b}{k_g}}(2k_b - k_g)k_vQ_d^2\left(\frac{2k_vQ_d}{A} + k_gz\right)^{\frac{2k_b}{k_g}}}{2k_ck_vQ_d^2\left(\frac{k_vQ_d}{A} + \frac{k_gz}{2}\right)^{\frac{2k_b}{k_g}} - \left(\frac{k_vQ_d}{A}\right)^{\frac{2k_b}{k_g}}\chi} \tag{66}$$

where

$$\chi = \left(A(-2k_b + k_g) + k_cQ_d\right)\left(2k_vQd + Ak_gz\right) \tag{67}$$

The first and the second moments are given by:

$$\mu_1(z) = \frac{2k_vQ_d + Ak_gz}{2Ak_v} \tag{68}$$

$$\mu_2(z) = \frac{\left(2k_vQ_d + Ak_gz\right)^2}{4A^2(2k_b - 3k_g)k_v^3}\left[6k_ck_v + \chi\left(\frac{\left(\frac{2k_vQ_d}{A}\right)}{\left(\frac{2k_vQ_d}{A} + k_gz\right)}\right)^{\frac{2k_b}{3k_g}}\right] \tag{69}$$

where

$$\chi = \frac{1}{Q_d^2}\left(2Ak_b - 3Ak_g - 3k_cQ_d\right)\left(2k_vQd + Ak_gz\right) \tag{70}$$

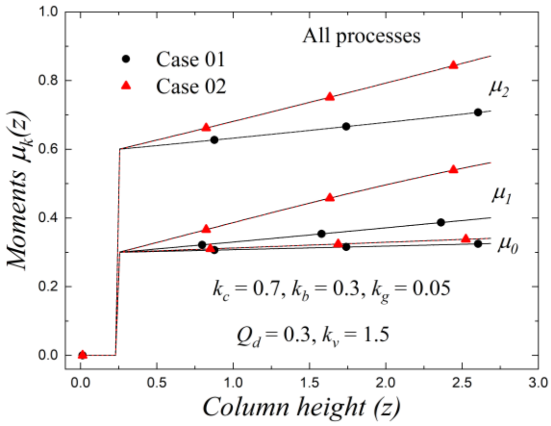

Fig.5. Spatially evolution of moments of the order 0,1 and 2 for simultaneous breakup aggregation and growth processes with two cases of the particle velocity.

After inserting $Q_d = 0.3, A = 1, k_v = 1.5, k_b = 0.3, k_c = 0.7$ and $k_g = 0.05$, analytical and numerical moments for both velocities are presented in the Figure 5.

The feed point of the dispersed phase and the nonactive zone appear clearly for $z \in [0, z_d]$ and $z = z_d$, respectively. All moments are directly proportional to the column height. For the first case, the moments in the active zone are similar to those in the batch system, but here, z is the independent variable instead of the time t. Comparatively, at the output, notable moments are produced by the second model than to the first, it makes a difference of 40.2% for μ_0, 4.7% for μ_1 and 22.6 for μ_2.

5 Conclusion

This study offers analytical solutions for a one-dimensional PBE model at steady-state that incorporates the processes of growth, breakup, and aggregation. The decomposition method and method of moments are used to solve the PBE analytically, ADM is tested for a pure breakup, pure aggregation, and simultaneous breakup with growth, and MOM for growth with breakup with aggregation. Successfully, exact solutions are found for all problems.

Since the proposed problems include different processes and different models of the particle velocity constant, space-dependent and volume-dependent, this study introduces knowledge to comprehend the dispersed phase behavior in the two-phase columns. In order to offer an analytical solution, ADM is an effective approach for solving the PBE. MOM can be used to reduce the PBE and overcomes the problems presented by the integrals.

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Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Nomenclature

A	column cross-section area
A_i	Adomian polynomial of degree i
B	daughter drop conditional probability distribution
c_v	form factor
D_d	dispersion coefficient
d	particle diameter
G	growth rate
g	breakage frequency
h	column height
H	aggrgation, breakup and growth source terms
k_b	breakage frequency model parameter
k_c	aggreagtion frequency model parameter
k_g	growth rate model parameter
k_d, k_v	particle velocity coeffecients
L	invertible operator
m	mean of the Gaussian distribution
n	particle number distribution
n_{in}	particle number distribution of the feed
N	non-linear term
Q_d	dispersed phase volumetric flowrate
R	the remainder of the linear operator
s	standard deviation of the Gaussian distribution
v	particle volume
v_d	dispersed phase velocity
z, x	height coordinates

z_d	dispersed feed inlet

Greek letters

β	particle daughter particle distribution
$\delta(.)$	Dirac delta function
μ_j	regular moment of order j
ω	coalescence frequency

Subscripts

a	aggregation
b	breakup
g	growth
in	inlet