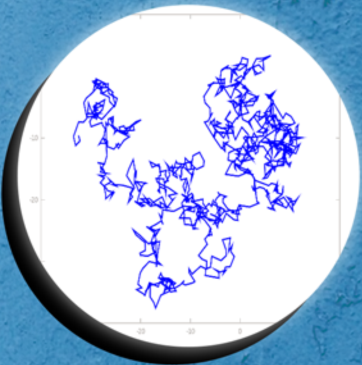
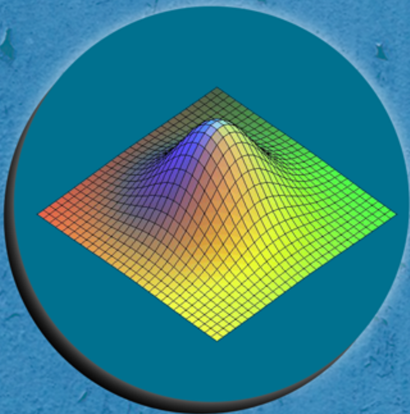
IJAMS

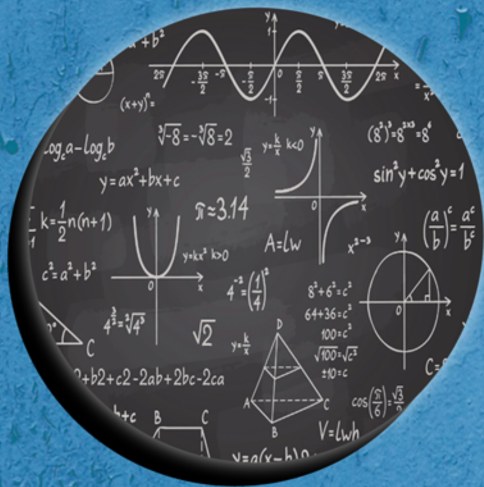
International Journal of Applied Mathematics and Simulation

Volume 2 **Issue 1** **2025**

ISSN (PRINT)=2992-1708 EISSN = 2992-1732







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Error estimation for a piezoelectric contact problem with wear and long memory

IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

Error estimation for a piezoelectric contact problem with wear and long memory

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DOI: https://doi.org/10.69717/ijams.v1.i2.107

abstract

We study a mathematical model for a quasistatic behavior of electro-viscoelastic materials. The problem is related to highly nonlinear and non-smooth phenomena like contact, friction and normal compliance with wear. Then, a fully discrete scheme is introduced based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. For a numerical scheme, we prove the existence and uniqueness of the solutions, and derive optimal order error estimates under certain regularity assumption on the solution of the continuous problem.

keywords

Quasistatic process, electro-viscoelastic materials, friction, wear, fully discrete scheme, error estimates.

2020 Mathematics Subject Classification

35J85 · 49J40 · 47J20 · 74M15

1. Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials. It consists of the appearance of electric charges on the surfaces of some crystals after their deformation. Conversely, experiments have shown that the action of an electric field on the crystals can generate stresses and deformations. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in Batra and Yang, 1995 and Ikeda, 1996. In Moumen and Rebiai, 2024 , the authors examine a transmission system of the Schrödinger equation with Neumann feedback control, which includes a time-varying delay term and acts on the exterior boundary. They utilize an appropriate energy function and a suitable Lyapunov functional. The authors of Acil et al., 2024 demonstrate the system's robustness, stability, and ability to respond to fast changes, making it a promising solution for efficient energy management in hybrid PV-battery systems. A static frictional contact problem for electric-elastic materials was considered in Maceri and Bisegna, 1998 and Migórski, 2006. Contact problems with friction or adhesion for electro-viscoelastic materials were studied in Selmani and Selmani, 2010 and Lerguet et al., 2007 and recently in Migórski et al., 2011 in the case of an electrically conductive foundation.

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Communicated Editor: T.J. RABEHERIMANAN
Manuscript received *Sep. 19, 2024;* ***revised*** *Feb 02, 2025;* ***accepted*** *Feb 11, 2025;* ***published*** *Feb 28, 2025..*

In this paper we consider a mathematical model for the process of contact with normal compliance and friction contact conditions when the wear of the contact surface due to friction is taken into account. The foundation is assumed to move steadily and only sliding contact takes places. The material is electro-viscoelastic with long memory, defined by a relaxation operator.

This work constitutes in some sense a continuation paper of the results obtained in Selmani, 2013. The work in Selmani, 2013 has been devoted to a qualitative results like existence and uniqueness result of weak solutions on displacement, electric potential and wear fields have been proved but no numerical approximations have been performed. Here we follow the latter work and propose a numerical scheme for the approximation of the solution fields so as to elaborate a general numerical analysis of error estimates.

The main goal of this work is to formulate an approximate solution of our problem, which can quickly converge to the exact solution. For that, this work is organized as follows. In Section 3 we give a short description of the mathematical model and recall the main existence and uniqueness result. In Section 4, For the numerical scheme, we prove the existence and uniqueness of the solutions. Finally, in Section 5, we derive optimal-order error estimates under certain regularity assumptions on the solution of the continuous problem.

2. Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\ .\ |$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces:

$$H = \left\{ \boldsymbol{u} = (u_i) \ / \ u_i \in L^2(\Omega) \right\},$$

$$\mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},$$

$$H_1 = \{ \boldsymbol{u} = (u_i) \ / \ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H} \},$$

$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \ / \ Div \boldsymbol{\sigma} \in H \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div \boldsymbol{\sigma} = (\sigma_{ij,\, j}).$$

A subscript that follows a comma indicates a partial derivative with respect to the corresponding spatial variable, e.g., $u_{i,j} = \partial u_i / \partial x_j$.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \ dx,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \ dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div \boldsymbol{\sigma}, Div \boldsymbol{\tau})_H .$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|.|_H$, $|.|_{\mathcal{H}}$, $|.|_{H_1}$ and $|.|_{\mathcal{H}_1}$, respectively. For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}.\boldsymbol{\nu}, \ \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{v}.\boldsymbol{v} da \ \forall \boldsymbol{v} \in H_1. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively. We use dots for derivatives with respect to the time variable t.

The space $C(0,T;X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X + \max_{t\in[0,T]} \left|\dot{\boldsymbol{f}}(t)\right|_X .$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Statement of the problem

An electro-viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface Γ. The body is subjected to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. We consider a partition of Γ into three disjoint measurable subsets Γ_1, Γ_2 and Γ_3, on one hand, and on two disjoint measurable subsets Γ_a and Γ_b, on the other hand, such that *meas* $(\Gamma_1) > 0$, *meas* $(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on Γ_1, so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface free electrical charge of density q_2 is prescribed on Γ_b. In the reference configuration, the body may come in contact over Γ_3 with a conductive obstacle, which is also called the foundation. The contact is frictional and is modeled with normal compliance, taking into account the wear of the contact surfaces. The foundation is assumed to move steadily and only sliding contact takes places. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected.

We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0,T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic, i.e., the inertial terms in the momentum balance equations are neglected. To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\boldsymbol{x} \in \Omega\cup\Gamma$ and $t \in [0,T]$. Then, the classical formulation of the mechanical problem of sliding frictional contact problem with normal compliance and wear may be stated as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$*, an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$*, an electric displacement field* $\boldsymbol{D} : \Omega \times [0,T] \to \mathbb{R}^d$ *and a* wear function $\zeta : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds \tag{3.1}$$

$$+\mathcal{E}^*\nabla\varphi(t) \text{ in } \Omega \times (0,T),$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{B}\nabla\varphi \ in\ \Omega \times (0,T), \tag{3.2}$$

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \ \ in\ \Omega \times (0,T), \tag{3.3}$$

$$div\boldsymbol{D} = q_0 \ in\ \Omega \times (0,T), \tag{3.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \ \ on\ \ \Gamma_1 \times (0,T), \tag{3.5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \ \ on\ \ \Gamma_2 \times (0,T), \tag{3.6}$$

$$\begin{cases} -\sigma_\nu = p_\nu\left(u_\nu - g - \zeta\right), \\ \ \ |\boldsymbol{\sigma}_\tau| = p_\tau\left(u_\nu - g - \zeta\right), \\ \boldsymbol{\sigma}_\tau = -\lambda\left(\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\right),\ \lambda \geq 0, \\ \dot{\zeta} = -k_0 v^* \sigma_\nu, \end{cases} \ on\ \ \Gamma_3 \times (0,T), \tag{3.7}$$

$$\varphi = 0 \ on\ \ \Gamma_a \times (0,T), \tag{3.8}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = q_2 \ on\ \ \Gamma_b \times (0,T), \tag{3.9}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \ \zeta(0) = 0 \ in\ \Omega. \tag{3.10}$$

Here, equations (3.1) – (3.2) represent the constitutive law for a piezoelectric material with long memory where $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{M}$ is a relaxation fourth order tensor. $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$is its transposed and $\boldsymbol{B}$ denotes the electric permittivity tensor. Equations (3.3) and (3.4) represent the equilibrium equations for the stress and electric-displacement fields. Equations (3.5) and (3.6) are the displacement-traction boundary conditions, respectively. (3.7) represents the condition with normal compliance, friction and wear where g represents the initial gap between the body and the foundation, $k_0 > 0$ is a wear coefficient and $\boldsymbol{v}^*$ is the tangential velocity of the foundation such that $v^* = |\boldsymbol{v}^*|$. Equations (3.8) and (3.9) represent the electric boundary conditions. In (3.10) $\boldsymbol{u}_0$ is the given initial displacement and $\zeta(0) = 0$ means that at the initial moment the body is not subject to any prior wear.

To obtain a variational formulation of the problem (3.1) – (3.10) we introduce the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1 \ / \ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $c_k > 0$ which depends only on Ω and Γ_1 such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v}|_{\mathcal{H}} \geq c_k \, |\boldsymbol{v}|_{H_1} \, \forall \boldsymbol{v} \in V.$$

On the space V we consider the inner product and the associated norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(v))_{\mathcal{H}}\,, \ \ |\boldsymbol{v}|_V = |\boldsymbol{\varepsilon}(\boldsymbol{v}|_{\mathcal{H}} \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.11}$$

It follows from Korn's inequality that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V. Therefore $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and (3.11), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \leq c_0 \, |\boldsymbol{v}|_V \ \ \forall \boldsymbol{v} \in V. \tag{3.12}$$

We also introduce the spaces.

$$W = \left\{\phi \in H^1(\Omega) \ / \ \phi = 0 \text{ on } \Gamma_a\right\},$$

$$\mathcal{W} = \left\{\boldsymbol{D} = (D_i) \ / \ D_i \in L^2(\Omega), \ div\boldsymbol{D} \in L^2(\Omega)\right\},$$

where $div\boldsymbol{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi . \nabla\phi \; dx, \tag{3.13}$$

$$(\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = \int_\Omega \boldsymbol{D}.\boldsymbol{E} \; dx + \int_\Omega div\boldsymbol{D}.div\boldsymbol{E} \; dx. \tag{3.14}$$

The associated norms will be denoted by $|.|_W$ and $|.|_{\mathcal{W}}$, respectively. Moreover, when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{D}, \nabla\phi)_H + (div\boldsymbol{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}.\boldsymbol{\nu} \; \phi da \ \ \forall \phi \in H^1(\Omega).$$

Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$|\nabla\phi|_H \geq c_F \, |\phi|_{H^1(\Omega)} \, \forall \phi \in W, \tag{3.15}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. It fallows from (3.15) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therfore $(W, |.|_W)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and (3.13), there exists a constant $a_0 > 0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\phi|_{L^2(\Gamma_3)} \leq a_0 \, |\phi|_W \ \ \forall \, \phi \in W. \tag{3.16}$$

In the study of the mechanical problem (3.1) – (3.10), we make the following assumptions. Assume that the operators $\mathcal{A}$, $\mathcal{F}$, $\mathcal{E}$, $\boldsymbol{B}$ and the functions p_r $(r = \nu, \tau)$ satisfy the following conditions with $L_{\mathcal{A}}$, $m_{\mathcal{A}}$, $L_{\mathcal{F}}$, L_r and m_r being positive constants:

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \\ \text{(b) } |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue} \\ \qquad \text{measurable in } \Omega \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{3.17}$$

$$\begin{cases} \text{(a) } \mathcal{F}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d \\ \text{(b) } |\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)|\le L_{\mathcal{F}}\,|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2| \\ \quad \forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c) The mapping } \boldsymbol{x}\to\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d. \\ \text{(d) } \boldsymbol{x}\to\mathcal{F}(\boldsymbol{x},\boldsymbol{0})\in\mathcal{H}. \end{cases} \tag{3.18}$$

$$\begin{cases} \text{(a) } \mathcal{E}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d \\ \text{(b) } \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}=(e_{ijk}(\boldsymbol{x})\tau_{jk}) \\ \quad \forall\boldsymbol{\tau}=(\tau_{ij})\in\mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c) } e_{ijk}=e_{ikj}\in L^\infty(\Omega). \end{cases} \tag{3.19}$$

$$\begin{cases} \text{(a) } \boldsymbol{B}=(b_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d \\ \text{(b) } \boldsymbol{B}(\boldsymbol{x})\boldsymbol{E}=(b_{ij}(x)E_j) \\ \quad \forall\boldsymbol{E}=(E_i)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(c) } b_{ij}=b_{ji},\ b_{ij}\in L^\infty(\Omega). \\ \text{(d) } \boldsymbol{BE}.\boldsymbol{E}\ge m_B\,|\boldsymbol{E}|^2 \\ \quad \forall\boldsymbol{E}=(E_i)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{3.20}$$

$$\begin{cases} \text{(a) } p_r:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+\ (r=\nu,\tau) \\ \text{(b) } |p_r(\boldsymbol{x},\alpha_1)-p_r(\boldsymbol{x},\alpha_2)|\le L_r\,|\alpha_1-\alpha_2| \\ \quad \forall\alpha_1,\alpha_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \\ \text{(c) } |p_r(\boldsymbol{x},\alpha)|\le m_r\ \forall\alpha\in\mathbb{R}, \text{ p.p. } \boldsymbol{x}\in\Gamma_3, \\ \text{(d) The mapping } \boldsymbol{x}\to p_r(\boldsymbol{x},\alpha) \text{ is Lebesgue} \\ \quad \text{measurable on } \Gamma_3 \text{ for any } \alpha\in\mathbb{R}. \\ \text{(e) } \boldsymbol{x}\to p_r(\boldsymbol{x},0)\in L^2(\Gamma_3). \end{cases} \tag{3.21}$$

The relaxation tensor $\mathcal{M}$ satisfies

$$\mathcal{M}\in C(0,T;\mathcal{H}_\infty), \tag{3.22}$$

where $\mathcal{H}_\infty$ is the space of fourth order tensor field given by

$$\mathcal{H}_\infty=\{\boldsymbol{E}=(E_{ijkl})\ /\ E_{ijkl}=E_{jikl}=E_{klij}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d\},$$

which is a real Banach space with the norm

$$|\boldsymbol{E}|_{\mathcal{H}_\infty}=\sum_{1\le i,j,k,l\le d}|E_{ijkl}|_{L^\infty(\Omega)}.$$

The density of volume forces, traction, volume electric charges and surface electric charges have the regularity

$$\boldsymbol{f}_0\in C(0,T;H),\ \boldsymbol{f}_2\in C(0,T;L^2(\Gamma_2)^d), \tag{3.23}$$

$$q_0\in C(0,T;L^2(\Omega)),\ q_2\in C(0,T;L^2(\Gamma_b)). \tag{3.24}$$

$$q_2=0 \text{ on } \Gamma_3\ \forall t\in[0,T]. \tag{3.25}$$

We assume that the gap function g and the initial displacement field $\boldsymbol{u}_0$ satisfy

$$g\in L^2(\Gamma_3),\ g\ge 0 \text{ a.e. } \boldsymbol{x}\in\Gamma_3. \tag{3.26}$$

$$\boldsymbol{u}_0\in V. \tag{3.27}$$

We define the three mappings $\boldsymbol{f}:[0,T]\to V$, $q:[0,T]\to W$ and $j:V\times V\times L^2(\Gamma_3)\to\mathbb{R}$, respectively, by

$$(\boldsymbol{f}(t),\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_0(t).\boldsymbol{v}dx+\int_{\Gamma_2}\boldsymbol{f}_2(t).\boldsymbol{v}da, \tag{3.28}$$

$$(q(t),\phi)_W=\int_\Omega q_0(t)\phi dx-\int_{\Gamma_b}q_2(t)\phi da. \tag{3.29}$$

$$j(\boldsymbol{u},\boldsymbol{v},\zeta)=\int_{\Gamma_3}p_\nu(u_\nu-g-\zeta)v_\nu da \tag{3.30}$$

$$+\int_{\Gamma_3}p_\tau(u_\nu-g-\zeta)\,|\boldsymbol{v}_\tau-\boldsymbol{v}^*|\,da,$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\zeta \in L^2(\Gamma_3)$ and $t \in [0,T]$. The functional $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} \text{For all } \boldsymbol{u} \in V \text{ and } \zeta \in L^2(\Gamma_3),\ \boldsymbol{v} \to j(\boldsymbol{u}, \boldsymbol{v}, \zeta) \\ \text{is proper, convex and lower semicontinuous on } V. \end{array} \right.$$

We note that condition (3.23) and (3.24) imply that

$$\boldsymbol{f} \in C(0,T;V),\ q \in C(0,T;W). \tag{3.31}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1) – (3.10).

Problem VP. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$, *an electric potential field* $\varphi : [0,T] \to W$, *an electric displacement field* $\boldsymbol{D} : [0,T] \to \mathcal{W}$ *and a wear function* $\zeta : [0,T] \to L^2(\Gamma_3)$ *such that for all* $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{M}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds + \mathcal{E}^*\nabla\varphi(t), \tag{3.32}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{v}, \zeta(t)) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \zeta(t)) \tag{3.33}$$

$$\geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V,$$

$$\boldsymbol{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{B}\nabla\varphi(t), \tag{3.34}$$

$$(\boldsymbol{D}(t), \nabla\phi)_H = -(q(t), \phi)_W \quad \forall \phi \in W, \tag{3.35}$$

$$\dot{\zeta} = k_0 \upsilon^* p_\nu (u_\nu - g - \zeta), \tag{3.36}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \zeta(0) = 0, \tag{3.37}$$

The main result in this section is the following existence and uniqueness result (see for details Selmani, 2013).

Theorem 3.1. *Assume that* (3.17) – (3.27) *hold. Then, there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \zeta\}$ *to Problem VP. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{3.38}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{3.39}$$

$$\varphi \in C(0,T;W), \tag{3.40}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}), \tag{3.41}$$

$$\zeta \in C^1(0,T;L^2(\Gamma_3)). \tag{3.42}$$

4. Fully discrete approximation

In this section, we introduce a discrete numerical scheme of Problem VP. We assume that the conditions (3.17) – (3.27) hold. Thus, it follows from Theorem 3.1 that Problem VP has a unique solution. More precisely, we are interested in solving Problem VP over a finite time interval $[0,T]$, with $T > 0$ arbitrary but fixed. Thus, let N be a positive integer; we define the time step size $k = \frac{T}{N}$ and we consider the uniform time discretization $t_n = nk$, $0 \leq n \leq N$, where N is a sufficiently large integer. For a continuous function $\upsilon(t)$ with values in a function space, we write $\upsilon_j = \upsilon(t_j)$, $0 \leq j \leq N$. For spatial discretization, we consider a polygonal domain Ω.

For the discretization of the integrals, we use the rectangle method

$$\int_{t_j}^{t_{j+1}} \upsilon(s)ds = k\upsilon_j.$$

Let $\mathcal{H}^h$ and B^h be the finite element spaces of piecewise constants. The spaces $\mathcal{H}$ and $L^2(\Gamma_3)$ are approximated by $\mathcal{H}^h$ and B^h, respectively.

The V and W spaces are approximated respectively by the following finite element spaces:

$$V^h = \left\{ \boldsymbol{v}^h \in \left[C\left(\overline{\Omega}\right)\right]^d \mid \boldsymbol{v}^h|_K \in \left[P_1(K)\right]^d\ \forall K \in \mathcal{T}_h,\ \boldsymbol{v}^h = 0 \text{ on } \Gamma_1 \right\},$$

$$W^h = \left\{ \phi^h \in C\left(\overline{\Omega}\right) \mid \phi^h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h,\ \phi^h = 0 \text{ on } \Gamma_a \right\},$$

where $\mathcal{T}_h$ is an element derived from the triangularization of $\overline{\Omega}$, $P_1(K)$ is the space of polynomials of degree smaller or equal to one on K and h refers to the spatial discretion parameter which is defined as

$$h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K), \text{with } \operatorname{diam}(K) = \max\{|x-y|;\ x, y \in K\}.$$

For all $\boldsymbol{\tau} \in \mathcal{H}$, $\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}$ is the orthogonal projection of finite elements on $\mathcal{H}^h$,

$$\left(\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} = \left(\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} \ \ \forall \boldsymbol{\tau}^h \in \mathcal{H}^h.$$

It is convenient to introduce the velocity field

$$\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t) \text{ so } \boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)ds, t \in [0,T].$$

It follows from Theorem 3.1 that $\boldsymbol{v} \in C(0,T;V)$ and for all $t \in [0,T]$, we have

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{M}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds + \mathcal{E}^*\nabla\varphi(t), \tag{4.1}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{v}, \zeta(t)) - j(\boldsymbol{u}(t), \boldsymbol{v}(t), \zeta(t)) \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}(t))_V \ \ \forall \boldsymbol{v} \in V,\end{aligned} \tag{4.2}$$

Let $\boldsymbol{u}_0^h \in V^h$ be a finite element approximation of $\boldsymbol{u}_0$.

The fully discrete approximation of Problem VP is the following.

Problem VP^{hk}. *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \left\{\boldsymbol{v}_n^{hk}\right\}_{n=0}^N \subset V^h$, *a discrete stress field* $\boldsymbol{\sigma}^{hk} = \left\{\boldsymbol{\sigma}_n^{hk}\right\}_{n=0}^N \subset \mathcal{H}^h$, *a discrete electric potential* $\varphi^{hk} = \left\{\varphi_n^{hk}\right\}_{n=0}^N \subset W^h$ *and a discrete wear field* $\zeta^{hk} = \left\{\zeta_n^{hk}\right\}_{n=0}^N \subset B^h$ *such that*

$$\boldsymbol{\sigma}_0^h = \mathcal{P}_{\mathcal{H}^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h) + \mathcal{P}_{\mathcal{H}^h}\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h) + \mathcal{P}_{\mathcal{H}^h}\mathcal{E}^*\nabla\varphi_0^h, \tag{4.3}$$

$$\begin{aligned}(\boldsymbol{\sigma}_0^h, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_0^h))_{\mathcal{H}} + j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0) \\ \geq (\boldsymbol{f}(0), \boldsymbol{v}^h - \boldsymbol{v}_0^h)_V \ \ \forall \boldsymbol{v}^h \in V^h,\end{aligned} \tag{4.4}$$

$$\begin{aligned}(\boldsymbol{B}\nabla\varphi_0^h, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\phi^h)_H \\ = (q(0), \phi^h)_W \ \ \forall \phi^h \in W^h,\end{aligned} \tag{4.5}$$

and for $n \geq 1$,

$$\begin{aligned}\boldsymbol{\sigma}_n^{hk} = \mathcal{P}_{\mathcal{H}^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + \mathcal{P}_{\mathcal{H}^h}\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \mathcal{P}_{\mathcal{H}^h}\mathcal{E}^*\nabla\varphi_n^{hk} \\ +k\sum_{j=0}^{n-1}\mathcal{P}_{\mathcal{H}^h}\left(\mathcal{R}_n\right)_j^{hk}\end{aligned} \tag{4.6}$$

$$\begin{aligned}(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk}) \\ \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{v}_n^{hk})_V \ \ \forall \boldsymbol{v}^h \in V^h,\end{aligned} \tag{4.7}$$

$$\begin{aligned}(\boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\phi^h)_H \\ = (q_n, \phi^h)_W \ \ \forall \phi^h \in W^h,\end{aligned} \tag{4.8}$$

$$\zeta_n^{hk} = kk_0\upsilon^*\sum_{j=0}^{n-1}p_\nu\left(u_{\nu j}^{hk} - g - \zeta_j^{hk}\right). \tag{4.9}$$

Here, we used the following notations

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h, \ \boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h, \ \boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h, \ \varphi_0^{hk} = \varphi_0^h \text{ and } \zeta_0^{hk} = \zeta_0^h = \zeta_0 = 0.$$

We use the following discrete displacement field

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + k\sum_{j=1}^{n}\boldsymbol{v}_j^{hk} \ n \geq 1, \tag{4.10}$$

We also use the notations

$$\begin{cases}\left(\mathcal{R}_n\right)_j^{hk} = \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}), \\ \left(\mathcal{R}_n\right)(s) = \mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \\ \left(\mathcal{R}_n\right)_j = \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j).\end{cases} \tag{4.11}$$

We have the following existence and uniqueness result.

Theorem 4.1. *Suppose that the conditions stated in Theorem 3.1 are satisfied. Then the Problem VP*^{hk} *has a unique solution.*

Proof. First,we show that (4.3) – (4.5) uniquely determines $\sigma_0^h \in \mathcal{H}^h$, $\upsilon_0^h \in V^h$ and $\varphi_0^h \in W^h$. From a discrete analogue of Lemma 4.5 in Selmani, 2013, it follows that (4.5) has a unique solution $\varphi_0^h \in W^h$. Combining (4.3) and (4.4), we obtain an elliptic variational inequality which has a unique solution $\boldsymbol{\upsilon}_0^h \in V^h$. $\boldsymbol{\sigma}_0^h \in \mathcal{H}^h$ is then calculated from (4.3).

Next, we show that with $\left\{\left(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{\upsilon}_j^{hk}, \varphi_j^{hk}, \zeta_j^{hk}\right)\right\}_{j\leq n-1} \subset \mathcal{H}^h \times V^h \times W^h \times B^h$ known, (4.6) – (4.9) uniquely determines $\left(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\upsilon}_n^{hk}, \varphi_n^{hk}, \zeta_n^{hk}\right) \subset \mathcal{H}^h \times V^h \times W^h \times B^h$. Given $\left\{\left(\boldsymbol{u}_j^{hk}, \zeta_j^{hk}\right)\right\}_{j\leq n-1} \in V^h \times B^h$, a discrete analogue of Lemma 4.5 in Selmani, 2013 shows that (4.8) has a unique solution $\varphi_n^{hk} \in W^h$ and $\zeta_n^{hk} \in B^h$ is computed from (4.9).

Finally, combining (4.6) and (4.7), we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h - \boldsymbol{\upsilon}_n^{hk}))_{\mathcal{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{\upsilon}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{\upsilon}_n^{hk}, \zeta_n^{hk}) \tag{4.12}$$
$$\geq (\boldsymbol{r}_n, \boldsymbol{\upsilon}^h - \boldsymbol{\upsilon}_n^{hk})_V \quad \forall \boldsymbol{\upsilon}^h \in V^h,$$

where

$$(\boldsymbol{r}_n, \boldsymbol{\upsilon}^h)_V = (\boldsymbol{f}_n, \boldsymbol{\upsilon}^h)_V - \Big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \mathcal{E}^*\nabla\varphi_n^{hk} \tag{4.13}$$
$$+k\sum_{j=0}^{n-1} (\mathcal{R}_n)_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h)\Big)_{\mathcal{H}}.$$

By a standard result on elliptic variational inequalities, there exists a unique $\boldsymbol{\upsilon}_n^{hk} \in V^h$ satisfying (4.12). We compute $\boldsymbol{\sigma}_n^{hk}$ from (4.6). □

5. Error estimates

This section is devoted to deriving error estimates for the discrete solution. We make the following solution regularity assumptions:

$$(\mathcal{M}, \boldsymbol{u}, \zeta) \in C^1(0, T; \mathcal{H}_\infty \times V \times L^2(\Gamma_3)), \tag{5.1}$$
$$(\boldsymbol{\upsilon}, \boldsymbol{\sigma}, \varphi) \in C(0, T; V \times \mathcal{H}_1 \times W), \tag{5.2}$$
$$(\boldsymbol{\upsilon}, \boldsymbol{\sigma}, \varphi) \in C(0, T; H^2(\Omega)^d \times H^1(\Omega)^{d\times d} \times H^2(\Omega)), \tag{5.3}$$
$$\boldsymbol{\upsilon} \in C\left(0, T; H^2(\Gamma_3)^d\right), \boldsymbol{\sigma\nu} \in C(0, T; L^2(\Gamma)^d), \boldsymbol{u}_0 \in H^2(\Omega)^d. \tag{5.4}$$

In this section, no summation is assumed over a repeated index and c denotes a positive constant which depends on the problem data, but is independent on the discretization parameters, h and k.

Lemma 5.1. *Assume that* (3.17) – (3.27) *hold. Let* $\{\boldsymbol{\sigma}, \boldsymbol{\upsilon}, \boldsymbol{u}, \varphi, \zeta\}$ *and* $\left\{\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\upsilon}_n^{hk}, \boldsymbol{u}_n^{hk}, \varphi_n^{hk}, \zeta_n^{hk}\right\}$ *denote the solution to Problems* **VP** *and* $\mathbf{VP}^{hk}$*, respectively. Then, the following error estimates hold for all* $\boldsymbol{\upsilon}^h \in V^h$ *and* $\phi^h \in W^h$ *:*

$$\max_{0\leq n\leq N}\left\{\begin{array}{c} \left|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}} + \left|\boldsymbol{\upsilon}_n - \boldsymbol{\upsilon}_n^{hk}\right|_V + \left|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right|_V \\ + \left|\varphi_n - \varphi_n^{hk}\right|_W + \left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} \end{array}\right\} \tag{5.5}$$
$$\leq ck + c\Big\{\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V + \max_{0\leq n\leq N}\left|(I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}} + \max_{0\leq n\leq N}\inf_{\phi^h\in W^h}\left|\varphi_n - \phi^h\right|_W$$
$$+ \max_{0\leq n\leq N}\inf_{\boldsymbol{\upsilon}^h\in V^h}\left(\left|\boldsymbol{\upsilon}_n - \boldsymbol{\upsilon}^h\right|_V + \left|\boldsymbol{\upsilon}_n - \boldsymbol{\upsilon}^h\right|_{L^2(\Gamma_3)^d}^{\frac{1}{2}}\right)\Big\}.$$

Proof. First, we make an error estimate on the electric potential. We combine (3.34) and (3.35), we have for all $t \in [0, T]$ and $\phi \in W$,

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\phi)_H = (q(t), \phi)_W. \tag{5.6}$$

Taking (5.6) at $t = t_n$ and for all $\phi = \phi^h \in W^h$ and $n \geq 1$, it follows that

$$(\boldsymbol{B}\nabla\varphi_n, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \nabla\phi^h)_H = (q(t), \phi^h)_W. \tag{5.7}$$

We subtract (4.8) from (5.7) to obtain for all $\phi^h \in W^h$ and $n \geq 1$

$$(\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\phi^h)_H = 0,$$

thus

$$(\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\phi^h - \varphi_n^{hk}\right))_H = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\phi^h - \varphi_n^{hk}\right))_H,$$

using the writing $\phi^h = \phi^h + \varphi_n - \varphi_n$, we see that

$$\begin{aligned} & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ = & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^{h}\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \phi^{h}\right))_H. \end{aligned}$$

Using (3.20) to see that

$$\begin{aligned} m_B \left|\varphi_n - \varphi_n^{hk}\right|_W^2 \leq & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^{h}\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \phi^{h}\right))_H, \end{aligned}$$

using the Cauchy-Schwarz inequality and the following inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \ \forall \epsilon > 0, \tag{5.8}$$

we obtain

$$\left|\varphi_n - \varphi_n^{hk}\right|_W^2 \leq c\left(\left|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right|_V^2 + \left|\varphi_n - \phi^h\right|_W^2\right). \tag{5.9}$$

From (5.6) at $t = 0$ with $\phi = \phi^h \in W^h$, we have

$$(\boldsymbol{B}\nabla\varphi_0, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \nabla\phi^h)_H = (q\left(0\right), \phi^h)_W,$$

We subtract (4.5) from the previous equality to obtain

$$(\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\phi^h)_H = 0,$$

then, we can write

$$(\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\phi^h - \varphi_0^h\right))_H = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\phi^h - \varphi_0^h\right))_H.$$

We use the writing $\phi^h = \phi^h - \varphi_0 + \varphi_0$ to note

$$\begin{aligned} & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ = & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \phi^h\right))_H. \end{aligned}$$

By using (3.20) to see that

$$\begin{aligned} m_B \left|\varphi_0 - \varphi_0^h\right|_W^2 \leq & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \phi^h\right))_H, \end{aligned}$$

Using the inequality of Cauchy-Schwarz, (3.19) – (3.20) and (5.8), we find

$$\left|\varphi_0 - \varphi_0^h\right|_W^2 \leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \phi^h\right|_W^2\right). \tag{5.10}$$

Next, we state two relations that we will use in error estimations (see Sofonea et al., 2005)

$$\left|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right|_V^2 \leq ck^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + ck\sum_{j=1}^{n}\left|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right|_V^2, \tag{5.11}$$

$$\left|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right|_V^2 \leq ck^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + ck\sum_{j=0}^{n-1}\left|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right|_V^2. \tag{5.12}$$

We note for all $n \geq 1$

$$\begin{aligned}\theta_n^{hk}(\mathcal{R}_n) &= \int_0^{t_n} (\mathcal{R}_n)(s)ds - \sum_{j=0}^{n-1} k(\mathcal{R}_n)_j^{hk} \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[(\mathcal{R}_n)(s) - (\mathcal{R}_n)_j\right] ds \\ &\quad + \sum_{j=0}^{n-1} k\left[(\mathcal{R}_n)_j - (\mathcal{R}_n)_j^{hk}\right],\end{aligned}$$

then

$$\theta_n^{hk}(\mathcal{R}_n) = I_n + I_n^{hk}, \tag{5.13}$$

where

$$I_n = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[(\mathcal{R}_n)(s) - (\mathcal{R}_n)_j\right] ds,\ I_n^{hk} = \sum_{j=0}^{n-1} k\left[(\mathcal{R}_n)_j - (\mathcal{R}_n)_j^{hk}\right].$$

We have

$$\begin{aligned}I_n &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [\mathcal{M}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \mathcal{M}(t_n-t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j)]\, ds \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [\mathcal{M}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \mathcal{M}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_j)]\, ds \\ &\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} [\mathcal{M}(t_n-s) - \mathcal{M}(t_n-t_j)]\,\boldsymbol{\varepsilon}(\boldsymbol{u}_j)ds.\end{aligned}$$

We use the hypothesis (3.22), we obtain

$$|I_n|_{\mathcal{H}} \leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[|\boldsymbol{u}(s) - \boldsymbol{u}_j|_V + |\mathcal{M}(t_n-s) - \mathcal{M}(t_n-t_j)|_{\mathcal{H}_\infty}\right] ds.$$

Using (5.1), the sum can be bounded by ck where the constant c is proportional to $|\dot{\boldsymbol{u}}|_{C(0,T;V)} + \left|\dot{\mathcal{M}}\right|_{C(0,T;\mathcal{H}_\infty)}$. Hence

$$|I_n|_{\mathcal{H}}^2 \leq ck^2. \tag{5.14}$$

We also have

$$I_n^{hk} = \sum_{j=0}^{n-1} k\left[\mathcal{M}(t_n-t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j) - \mathcal{M}(t_n-t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})\right],$$

From (3.22) and (3.11), we find

$$\left|I_n^{hk}\right|_{\mathcal{H}} \leq ck\sum_{j=0}^{n-1} \left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V. \tag{5.15}$$

We combine (5.13) – (5.14) and (5.15) to see that

$$\left|\theta_n^{hk}(\mathcal{R}_n)\right|_{\mathcal{H}}^2 \leq ck^2 + ck\sum_{j=0}^{n-1} \left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V^2. \tag{5.16}$$

Furthermore, we apply (4.1) at $t = t_n$ to see that

$$\boldsymbol{\sigma}_n = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + \int_0^{t_n} \mathcal{M}(t_n-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds + \mathcal{E}^*\nabla\varphi_n. \tag{5.17}$$

Using (4.6) and (5.17), we can write for all $n \geq 1$

$$\begin{aligned}\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} &= (I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n + \mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} \\ &= (I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n + \mathcal{P}_{\mathcal{H}^h}\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\right) + \left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})\right)\right.\end{aligned}$$

$$+\left(\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}\right) + \theta_n^{hk}\left(\mathcal{R}_n\right)\right].$$

Here, we used the symbol I for the identity application on $\mathcal{H}$. using the hypotheses on operators $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{E}$, as well as inequality $|\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}|_{\mathcal{H}} \leq |\boldsymbol{\tau}|_{\mathcal{H}}$, we have

$$\left|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2 \leq c\left[\left|\left(I - \mathcal{P}_{\mathcal{H}^h}\right)\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right|_V^2\right] \tag{5.18}$$

$$+c\left[\left|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right|_V^2 + \left|\varphi_n - \varphi_n^{hk}\right|_W^2 + \left|\theta_n^{hk}\left(\mathcal{R}_n\right)\right|_{\mathcal{H}}^2\right].$$

For $n = 0$, using (4.1) at $t = 0$ and (4.3), we have

$$\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h = \left(I - \mathcal{P}_{\mathcal{H}^h}\right)\boldsymbol{\sigma}_0 + \mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h$$

$$= \left(I - \mathcal{P}_{\mathcal{H}^h}\right)\boldsymbol{\sigma}_0 + \mathcal{P}_{\mathcal{H}^h}\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h)\right) + \left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\right)\right.$$

$$\left.+\left(\mathcal{E}^*\nabla\varphi_0 - \mathcal{E}^*\nabla\varphi_0^h\right)\right].$$

Using (3.17) – (3.19) we find

$$\left|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\right|_{\mathcal{H}}^2 \leq c\left[\left|\left(I - \mathcal{P}_{\mathcal{H}^h}\right)\boldsymbol{\sigma}_0\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2\right]. \tag{5.19}$$

$$+c\left[\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \varphi_0^h\right|_W^2\right]$$

We combine (4.1) and (4.2), taking $t = t_n$ for all $\boldsymbol{v} \in V$ and $n \geq 1$, we obtain

$$\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + \int_0^{t_n}\left(\mathcal{R}_n\right)(s)ds + \mathcal{E}^*\nabla\varphi_n, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_n)\right)_{\mathcal{H}} \tag{5.20}$$

$$+j(\boldsymbol{u}_n, \boldsymbol{v}, \zeta_n) - j(\boldsymbol{u}_n, \boldsymbol{v}_n, \zeta_n) \geq (\boldsymbol{f}_n, \boldsymbol{v} - \boldsymbol{v}_n)_V.$$

By combining (4.6) and (4.7) to write for all $\boldsymbol{v}^h \in V^h$ and $n \geq 1$

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \mathcal{E}^*\nabla\varphi_n^{hk} + k\sum_{j=0}^{n-1}\left(\mathcal{R}_n\right)_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} \tag{5.21}$$

$$+j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk}) \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{v}_n^{hk})_V.$$

From (3.17) the hypothesis on $\mathcal{A}$, we have for all $n \geq 1$

$$\begin{aligned} m_{\mathcal{A}}\left|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right|_V^2 &\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} \\ &= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} \\ &\quad -(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h))_{\mathcal{H}} \\ &\quad +(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk} - \boldsymbol{v}^h))_{\mathcal{H}}. \end{aligned}$$

We use (5.20) with $\boldsymbol{v} = \boldsymbol{v}_n^{hk}$ to estimate the first term and (5.21) to estimate the third term, we add $(\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h))_{\mathcal{H}} - (\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h))_{\mathcal{H}}$ to the second side, after some elementary algebraic operations, we obtain

$$m_{\mathcal{A}}\left|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right|_V^2 \tag{5.22}$$

$$\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h))_{\mathcal{H}} + \left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \theta_n^{hk}\left(\mathcal{R}_n\right)\right.$$

$$\left.+\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h)\right)_{\mathcal{H}} - \left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \theta_n^{hk}\left(\mathcal{R}_n\right)\right.$$

$$\left.+\mathcal{E}^*\nabla\varphi_n - \mathcal{E}^*\nabla\varphi_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk})\right)_{\mathcal{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n, \zeta_n^{hk})$$

$$+j(\boldsymbol{u}_n, \boldsymbol{v}_n^{hk}, \zeta_n) - j(\boldsymbol{u}_n, \boldsymbol{v}_n, \zeta_n) + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk}) + \mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right),$$

where

$$\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right) = -(\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}_n - \boldsymbol{v}^h))_{\mathcal{H}} + (\boldsymbol{f}_n, \boldsymbol{v}_n - \boldsymbol{v}^h)_V. \tag{5.23}$$

From (3.30) the definition of j, we have for all $n \geq 1$

$$\left|j(\boldsymbol{u}_n, \boldsymbol{v}_n^{hk}, \zeta_n) - j(\boldsymbol{u}_n, \boldsymbol{v}_n, \zeta_n) + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk})\right|$$

$$= \left|\int_{\Gamma_3} p_\nu(u_{n\nu} - g - \zeta_n\ )v_{n\nu}^{hk}da + \int_{\Gamma_3} p_\tau(u_{n\nu} - g - \zeta_n\ )\left|\boldsymbol{v}_{n\tau}^{hk} - \boldsymbol{v}^*\right|da\right.$$

$$
\begin{aligned}
&-\int_{\Gamma_3} p_\nu(u_{n\nu}-g-\zeta_n\ )\upsilon_{n\nu}da-\int_{\Gamma_3} p_\tau(u_{n\nu}-g-\zeta_n\ )\left|\boldsymbol{v}_{n\tau}-\boldsymbol{v}^*\right|da\\
&+\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon_{n\nu}da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{v}_{n\tau}-\boldsymbol{v}^*\right|da\\
&-\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon^{hk}_{n\nu}da-\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{v}^{hk}_{n\tau}-\boldsymbol{v}^*\right|da\Big|\\
&=\Big|\int_{\Gamma_3}\left[p_\nu(u_{n\nu}-g-\zeta_n\ )-p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right]\left[\upsilon^{hk}_{n\nu}-\upsilon_{n\nu}\right]da\\
&+\int_{\Gamma_3}\left[p_\tau(u_{n\nu}-g-\zeta_n\ )-p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right]\left[\left|\boldsymbol{v}^{hk}_{n\tau}-\boldsymbol{v}^*\right|-\left|\boldsymbol{v}_{n\tau}-\boldsymbol{v}^*\right|\right]da\Big|\\
&\le\int_{\Gamma_3}\left|p_\nu(u_{n\nu}-g-\zeta_n\ )-p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right|\left|\upsilon^{hk}_{n\nu}-\upsilon_{n\nu}\right|da\\
&+\int_{\Gamma_3}\left|p_\tau(u_{n\nu}-g-\zeta_n\ )-p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right|\left|\boldsymbol{v}^{hk}_{n\tau}-\boldsymbol{v}_{n\tau}\right|da.
\end{aligned}
$$

From (3.21) and inequality (3.12) with the inequality $|u_r|\le|\boldsymbol{u}|$ $(r=\nu,\tau)$ $\forall\boldsymbol{u}\in\mathbb{R}^d$, we find for all $n\ge1$

$$
\begin{aligned}
&\left|j(\boldsymbol{u}_n,\boldsymbol{v}^{hk}_n,\zeta_n)-j(\boldsymbol{u}_n,\boldsymbol{v}_n,\zeta_n)+j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}_n,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}^{hk}_n,\zeta^{hk}_n)\right| \qquad (5.24)\\
&\le(L_\nu+L_\tau)\,c_0^2\left|\boldsymbol{u}_n-\boldsymbol{u}^{hk}_{n-1}\right|_V\left|\boldsymbol{v}_n-\boldsymbol{v}^{hk}_n\right|_V\\
&+(L_\nu+L_\tau)\,c_0\left|\zeta_n-\zeta^{hk}_n\right|_{L^2(\Gamma_3)}\left|\boldsymbol{v}_n-\boldsymbol{v}^{hk}_n\right|_V.
\end{aligned}
$$

Similarly, we have for all $n\ge1$

$$
\begin{aligned}
&\left|j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}^h,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}_n,\zeta^{hk}_n)\right|\\
&=\Big|\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon^h_\nu da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{v}^h_\tau-\boldsymbol{v}^*\right|da\\
&-\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon_{n\nu}da-\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{v}_{n\tau}-\boldsymbol{v}^*\right|da\Big|\\
&\le\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\upsilon^h_\nu-\upsilon_{n\nu}\right|da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{v}^h_\tau-\boldsymbol{v}_{n\tau}\right|da
\end{aligned}
$$

Using (3.21) and (3.12) to deduce that

$$
\left|j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}^h,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{v}_n,\zeta^{hk}_n)\right|\le(m_\nu+m_\tau)\,c_0\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|^2_V. \qquad (5.25)
$$

We substitute (5.24) – (5.25) into (5.22) and using the assumptions on $\mathcal{A}$, $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{E}$, the Cauchy-Schwarz inequality and (5.8), we obtain for all $n\ge1$

$$
\begin{aligned}
&\left|\boldsymbol{v}_n-\boldsymbol{v}^{hk}_n\right|^2_V\le c\left(\left|\boldsymbol{u}_n-\boldsymbol{u}^{hk}_{n-1}\right|^2_V+\left|\varphi_n-\varphi^{hk}_n\right|^2_W+\left|\zeta_n-\zeta^{hk}_n\right|^2_{L^2(\Gamma_3)}\right) \qquad (5.26)\\
&+c\left(\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|^2_V+\left|\theta^{hk}_n\left(\mathcal{R}_n\right)\right|^2_{\mathcal{H}}\right)+\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|.
\end{aligned}
$$

Similarly, we apply (4.1) – (4.2) at $t=0$ with the initial condition $\zeta(0)=0$, for all $\boldsymbol{v}\in V$, we find

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0)+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)+\mathcal{E}^*\nabla\varphi_0,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{v}_0))_{\mathcal{H}} \qquad (5.27)\\
&+j(\boldsymbol{u}_0,\boldsymbol{v},0)-j(\boldsymbol{u}_0,\boldsymbol{v}_0,0)\ge(\boldsymbol{f}(0),\boldsymbol{v}-\boldsymbol{v}_0)_V.
\end{aligned}
$$

Using (4.3) – (4.4) with $\zeta^h_0=0$ to see that for all $\boldsymbol{v}^h\in V^h$

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^h_0)+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0)+\mathcal{E}^*\nabla\varphi^h_0,\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{v}^h_0))_{\mathcal{H}} \qquad (5.28)\\
&+j(\boldsymbol{u}^h_0,\boldsymbol{v}^h,0)-j(\boldsymbol{u}^h_0,\boldsymbol{v}^h_0,0)\ge(\boldsymbol{f}(0),\boldsymbol{v}^h-\boldsymbol{v}^h_0)_V.
\end{aligned}
$$

We use (3.17), we have

$$
\begin{aligned}
m_{\mathcal{A}}\left|\boldsymbol{v}_0-\boldsymbol{v}^h_0\right|^2_V &\le (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{v}_0-\boldsymbol{v}^{hk}_0))_{\mathcal{H}}\\
&= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0),\boldsymbol{\varepsilon}(\boldsymbol{v}_0-\boldsymbol{v}^h_0))_{\mathcal{H}}\\
&\quad-(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{v}_0-\boldsymbol{v}^h))_{\mathcal{H}}\\
&\quad+(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{v}^h_0-\boldsymbol{v}^h))_{\mathcal{H}}.
\end{aligned}
$$

Using (5.27) with $\boldsymbol{v} = \boldsymbol{v}_0^h$ to estimate the first term and (5.27) to estimate the third term, and adding $(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}} - (\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}}$ to the second side, we obtain

$$m_{\mathcal{A}} \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 \tag{5.29}$$

$$\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}} + \big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk})$$
$$+\mathcal{E}^*\nabla\varphi_0 - \mathcal{E}^*\nabla\varphi_0^h, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h)\big)_{\mathcal{H}} - \big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)$$
$$+\mathcal{E}^*\nabla\varphi_0 - \mathcal{E}^*\nabla\varphi_0^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}_0^h)\big)_{\mathcal{H}} + j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0)$$
$$+j(\boldsymbol{u}_0, \boldsymbol{v}_0^h, 0) - j(\boldsymbol{u}_0, \boldsymbol{v}_0, 0) + j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0) + \mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right).$$

From (3.21) and by the same argument that we used in (5.24), we find

$$\left|j(\boldsymbol{u}_0, \boldsymbol{v}_0^h, 0) - j(\boldsymbol{u}_0, \boldsymbol{v}_0, 0) + j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0)\right| \tag{5.30}$$
$$\leq (L_\nu + L_\tau)\, c_0^2 \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V.$$

Similarly, using a similar argument that we used in (5.25) to see that

$$\left|j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0)\right| \leq (m_\nu + m_\tau)\, c_0 \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2. \tag{5.31}$$

We substitute (5.30) – (5.31) into (5.29) and using (3.17) – (3.19), the Cauchy-Schwarz inequality and (5.8), we obtain

$$\left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 \leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \varphi_0^h\right|_W^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2\right) + \left|\mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right)\right|. \tag{5.32}$$

Combining (5.10) and (5.19) with (5.32), it is easy to see that

$$\left|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \varphi_0^h\right|_W^2 \tag{5.33}$$
$$\leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \phi^h\right|_W^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2 + \left|(I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_0\right|_{\mathcal{H}}^2\right) + \left|\mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right)\right|.$$

On the other hand, for the wear function, we use (3.36) at $t = t_n$, and $\zeta(0) = 0$, we obtain for all $n \geq 1$

$$\zeta_n = k_0 \upsilon^* \int_0^{t_n} p_\nu\left(u_\nu(s) - g - \zeta(s)\right) ds, \tag{5.34}$$

we subtract (4.9) from (5.34) to see that

$$\zeta_n - \zeta_n^{hk} = k_0 \upsilon^* \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(p_\nu\left(u_\nu(s) - g - \zeta(s)\right) - p_\nu\left(u_{\nu j}^{hk} - g - \zeta_j^{hk}\right)\right) ds\right],$$

using (3.21), the inequality $|u_\nu| \leq |\boldsymbol{u}|\ \forall \boldsymbol{u} \in \mathbb{R}^d$ and (3.12), we obtain

$$\begin{aligned}
\left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} &\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|u_\nu(s) - u_{\nu j}^{hk}\right|_{L^2(\Gamma_3)} + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds \\
&\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j^{hk}\right|_{L^2(\Gamma_3)^d} + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds \\
&\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j^{hk}\right|_V + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds,
\end{aligned}$$

therefore

$$\begin{aligned}
\left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} &\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j\right|_V + \left|\zeta(s) - \zeta_j\right|_{L^2(\Gamma_3)}\right] ds \\
&\quad + ck\sum_{j=0}^{n-1} \left[\left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V + \left|\zeta_j - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right],
\end{aligned}$$

using (5.1), the first sum can be bounded by ck where the constant c is proportional to $|\dot{\boldsymbol{u}}|_{C(0,T;V)}+\left|\dot{\zeta}\right|_{C(0,T;L^2(\Gamma_3))}$. Thus

$$\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2 \leq ck^2+ck\sum_{j=0}^{n-1}\left|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\right|_V^2+ck\sum_{j=0}^{n-1}\left|\zeta_j-\zeta_j^{hk}\right|_{L^2(\Gamma_3)}^2 \tag{5.35}$$

By adding (5.9), (5.11) – (5.12), (5.16), (5.18), (5.26) and (5.35) to obtain for all $n\geq 1$

$$\begin{aligned}
&\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V^2+\left|\varphi_n-\varphi_n^{hk}\right|_W^2+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2\\
&\leq ck^2+c\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V^2+c\left[\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2+\left|\varphi_n-\phi^h\right|_W^2+\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_V^2\right]\\
&\quad+\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|+ck\sum_{j=0}^{n-1}\left\{\left|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right|_V^2+\left|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\right|_V^2\right.\\
&\quad\left.+\left|\varphi_j-\varphi_j^{hk}\right|_W^2+\left|\zeta_j-\zeta_j^{hk}\right|_{L^2(\Gamma_3)}^2\right\}.
\end{aligned}$$

From this inegality and (5.33), applying Gronwall's Lemma (see for example Sofonea et al., 2012) to see that

$$\max_{0\leq n\leq N}\left\{\begin{array}{c}\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V^2\\ +\left|\varphi_n-\varphi_n^{hk}\right|_W^2+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2\end{array}\right\} \tag{5.36}$$

$$\begin{aligned}
&\leq ck^2+c\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V^2+c\max_{0\leq n\leq N}\left\{\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2+\inf_{\phi^h\in W^h}\left|\varphi_n-\phi^h\right|_W^2\right.\\
&\quad\left.+\inf_{\boldsymbol{v}^h\in V^h}\left[\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_V^2+\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\right]\right\}.
\end{aligned}$$

To find a bound of $\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)$ defined in (5.23), we integrate by parts the first term to obtain

$$\begin{aligned}
\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right) &= \int_\Omega Div\boldsymbol{\sigma}_n.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx-\int_\Gamma(\boldsymbol{\sigma}\boldsymbol{\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\
&\quad+(\boldsymbol{f}_n,\boldsymbol{v}_n-\boldsymbol{v}^h)_V.
\end{aligned}$$

Using (3.28) and we apply (3.3) and (3.6) at $t=t_n$ to see that for all $n\geq 0$

$$\begin{aligned}
\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right) &= -\int_\Omega \boldsymbol{f}_{0n}.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx-\int_{\Gamma_2}\boldsymbol{f}_{2n}\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\
&\quad-\int_{\Gamma_3}(\boldsymbol{\sigma}\boldsymbol{\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da+\int_\Omega \boldsymbol{f}_{0n}.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx\\
&\quad+\int_{\Gamma_2}\boldsymbol{f}_{2n}\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\
&= -\int_{\Gamma_3}(\boldsymbol{\sigma}\boldsymbol{\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da,
\end{aligned}$$

using the Cauchy-Schwarz inequality we see that

$$\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\leq\left|(\boldsymbol{\sigma}\boldsymbol{\nu})_n\right|_{L^2(\Gamma_3)^d}\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d}.$$

From (5.4) we deduce that

$$\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\leq c\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d},$$

Combining the previous estimate with (5.36), we find (5.5).□

Theorem 5.2. *Suppose that k is sufficiently small. Then, under the regularity assumptions* (5.1) – (5.4), *we have the following error estimate*

$$\max_{0\leq n\leq N}\left\{\begin{array}{c}\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V\\ +\left|\varphi_n-\varphi_n^{hk}\right|_W+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}\end{array}\right\}\leq c\,(h+k)\,. \tag{5.37}$$

Proof. Under assumptions (5.3) and (5.4), we can apply the standard theory of finite element interpolation (see for example Braess, 2007 and Sofonea et al., 2005) to see that

$$\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V\leq ch\left|\boldsymbol{u}_0\right|_{H^2(\Omega)^d},$$

$$\max_{0\le n\le N} |\boldsymbol{\sigma}_n - \mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_n|_{\mathcal{H}} \le ch\,|\boldsymbol{\sigma}|_{C\left(0,T;H^1(\Omega)^{d\times d}\right)},$$

$$\max_{0\le n\le N} \inf_{\boldsymbol{v}^h\in V^h} \left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_V \le ch\,|\boldsymbol{v}|_{C(0,T;H^2(\Omega)^d)},$$

$$\max_{0\le n\le N} \inf_{\boldsymbol{v}^h\in V^h} \left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d} \le ch^2\,|\boldsymbol{v}|_{C(0,T;H^2(\Gamma_3)^d)},$$

$$\max_{0\le n\le N} \inf_{\phi^h\in W^h} \left|\varphi_n - \phi^h\right|_W \le ch\,|\varphi|_{C(0,T;H^2(\Omega))}.$$

Combining the previous estimates and (5.5) it leads to (5.37).□

6. Conclusion

This paper presents a model of the quasistatic contact process between an electro-viscoelastic body and a foundation. The contact was modeled by normal compliance with wear. The proof of the existence of a unique weak solution to the model has been obtained by using arguments on elliptic variational inequalities. A fully discrete scheme is used to approach the problem and an optimal order error estimate. A numerical algorithm which combines the backward Euler difference method with the finite elements method. Finally, it may be interesting to incorporate control mechanisms into the model and study the related optimal control problem. Also, the problem is relatively easy to set experimentally, and it may provide an effective way to determine some of the constants associated with the contact process, to be used in more complex physical settings.

Acknowledgment

The author would like to thank to express their gratitude to the anonymous reviewers for their constructive feedback and the editor for their thorough evaluation of this manuscript.

Declaration

The author declares no conflicts of interest.

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The Impact of Imperfect Vaccination on Infectious Disease Transmission in an Age-Structured Population

IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

The Impact of Imperfect Vaccination on Infectious Disease Transmission in an Age-Structured Population

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DOI: https://doi.org/10.69717/ijams.v2.i1.104

abstract

In this paper, we consider the influence of imperfect vaccination on the spread of infectious diseases in an age-structured population. The benefits of vaccination, even if not perfect, generally outweigh the risks of severe diseases. In a mathematical system, we consider the compartment of susceptible s, vaccinated v and infected i individuals with an age structure.
The proposed model is globally analyzed by introducing total trajectories and employing a suitable Lyapunov functional. To illustrate our theoretical findings, we include numerical simulations at the end of the paper.

keywords

Age structured model; Lyapunov functional; Uniform persistence; Total trajectories; α and ω limit sets.

2020 Mathematics Subject Classification

35Q92, 37N25, 92D30

1. Introduction

Since its earliest applications, vaccination has been a highly effective strategy in preventing and controlling the spread of infectious diseases. This medical intervention not only plays a crucial role in individual and collective protection by stimulating the immune system against potentially devastating pathogens but has also been a subject of in-depth research, notably through mathematical modeling. This research aims to understand its impact on the dynamics of disease transmission within populations.

The importance of vaccination in containing the spread of infectious diseases cannot be overstated. Diseases like measles, polio, and influenza historically caused widespread devastation before the advent of vaccination programs. A notable success is the eradication of smallpox in the 1980s through a coordinated global vaccination campaign. These successes demonstrate that vaccination not only prevents disease in vaccinated individuals but also interrupts the chain of transmission, providing protection to unvaccinated populations, including vulnerable individuals who cannot be vaccinated for medical reasons.

The use of mathematical models in vaccination research has proved to be a valuable tool. It helps anticipate patterns of disease spread, assess the impact of vaccination campaigns, and design informed public health policies, we cite for example papers Adimy et al., 2022; Castillo-Chávez et al., 1989; Diekmann and Heesterbeek, 2000; Ismail and Touaoula, 2018.

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Communicated Editor: T.J. RABEHERIMANAN
Manuscript received *Sep. 19, 2024;* ***revised*** *Feb 02, 2025;* ***accepted*** *Feb 11, 2025;* ***published*** *Feb 28, 2025..*

While vaccines are recognized for reducing or eliminating infection rates, it's crucial to acknowledge that not all vaccines achieve 100% effectiveness Galazka et al., 1995; Grenfell and Anderson, 1989; Hethcote, 2000; Janaszek et al., 2003; Mossong et al., 2000; Scherer and McLean, 2002. Recent clinical studies have focused on understanding the impact of imperfect vaccines, characterized by waning or incomplete immunity, in controlling infectious disease transmission. These studies aim to answer key questions, including the proportion of susceptible individuals requiring immunization, the consequences of incomplete vaccine protection, and the significance of vaccine-induced immunity waning over time.

Vaccination programs offer both direct and indirect protection against infectious diseases **1**; Cai et al., 2013, 2017; Feng et al., 2020. Direct protection lowers the risk of infection in vaccinated individuals, while indirect protection limits transmission within populations. Various vaccine models, encompassing perfect and imperfect vaccines, have been explored, including all-or-nothing, leaky, and waning vaccines Mclean and Blower, 1993.

Numerous investigations have independently explored models of all-or-nothing, leaky, and waning vaccines for specific diseases. For example, Kanaan et al. devised a framework to examine the effectiveness of waning pertussis vaccines, demonstrating the potential to make inferences regarding the diminishing effects of these vaccines Kanaan and Farrington, 2002. Shim et al. employed dynamic epidemiological models for both all-or-nothing and leaky vaccines, emphasizing the critical role of accurately parameterizing vaccine effectiveness for robust model predictions Shim and Galvani, 2012. In a comprehensive study, Magpantay et al. delved into all-or-nothing, leaky, and waning vaccine models, investigating the variations in disease outcomes attributable to these different vaccine types Magpantay et al., 2014.

Studies have individually investigated these models for diseases like pertussis, measles, and rubella, with researchers assessing efficacy using dynamic epidemiological models. However, amidst the ongoing COVID-19 pandemic, vaccine prioritization discussions have primarily focused on all-or-nothing and leaky vaccine models, neglecting the consideration of waning vaccines Bubar et al., 2020; Buckner et al., 2020; Magpantay et al., 2014. This underscores the evolving challenges and the need for a comprehensive understanding of vaccine dynamics in the current global health landscape.

Imperfect vaccination can manifest in various forms, and here are some common types:

Partial Immunity: Some individuals may develop partial immunity after vaccination, meaning they are not entirely protected against the disease, but the severity of the infection can be reduced.

Limited Duration of Immunity: In some cases, immunity acquired through vaccination may decrease over time, eventually requiring regular boosters to maintain adequate protection.

Variable Effectiveness: The effectiveness of a vaccine can vary based on various factors such as age, the individual's overall health, and adherence to recommended vaccination schedules.

Protection Against Certain Serotypes: Some vaccines may offer protection against certain serotypes of pathogens, but not all. This can lead to infections by uncovered strains.

Rare Risks of Post-Vaccination Infection: While vaccines are designed to prevent infections, there may be rare cases where a vaccinated person still contracts the disease. However, the severity of the infection is often reduced in these individuals.

Viral Adaptation: Some viruses can undergo mutations over time, potentially reducing the effectiveness of vaccines against emerging strains. This may require regular adjustments to vaccine formulations.

Variable Immune Responses: Individuals may have different immune responses to vaccination due to genetic or environmental factors, leading to varying levels of protection.

It is important to note that, despite these imperfections, vaccination remains an essential tool for preventing and controlling the spread of infectious diseases. The benefits of vaccination, even if not perfect, generally outweigh the risks of severe diseases.

In Hathout et al., 2022, we considered a protective compartment, incorporating various aspects such as vaccination, within an SI model. The model construction implies that the protection (or vaccination) was perfect, as there was no transition of protected individuals to the infected compartment (perfect protection). The results are obtained based on the basic reproduction number $\mathcal{R}_0$. In this work, we will retain the same model while introducing an interaction between the i-class and v-class of the vaccinated. This implies that vaccination is not perfect:

$$
\begin{cases}
\dfrac{\partial s(t,a)}{\partial t}+\dfrac{\partial s(t,a)}{\partial a}=-(\mu_s(a)+\delta(a))s(t,a)-\beta_s(a)s(t,a)J(t), & t>0,\\
\dfrac{\partial v(t,a)}{\partial t}+\dfrac{\partial v(t,a)}{\partial a}=-(\mu_v(a)+k(a))v(t,a)-\beta_v(a)v(t,a)J(t), & t>0,\quad a>0,\\
\dfrac{\partial i(t,a)}{\partial t}+\dfrac{\partial i(t,a)}{\partial a}=-(\mu_i(a)+q(a))i(t,a), & t>0,\quad a>0,\\
s(t,0)=A+(1-\rho)\displaystyle\int_0^\infty k(a)v(t,a)da, & t>0,\\
v(t,0)=\displaystyle\int_0^\infty \delta(a)s(t,a)da+\rho\int_0^\infty k(a)v(t,a)da, & t>0,\\
i(t,0)=J(t)\displaystyle\int_0^\infty\left(\beta_s(a)s(t,a)+\beta_v(a)v(t,a)\right)da, & t>0,\\
J(t)=\displaystyle\int_0^\infty \theta(a)i(t,a)da,
\end{cases}
\tag{1}
$$

with initial conditions:

$$s(0,.)=\bar{s}(.)\in L^1_+(\mathbb{R}^+),\quad v(0,\cdot)=\bar{v}(\cdot)\in L^1_+(\mathbb{R}^+),\quad i(0,.)=\bar{i}(\cdot)\in L^1_+(\mathbb{R}^+).$$

$s(t,a)$, $v(t,a)$ and $i(t,a)$ are respectively the population densities of susceptible, protected and infected individuals, at time t with age a. Here, the age represents the time spent in each class. The functions $\mu_s(\cdot),\mu_v(\cdot)$

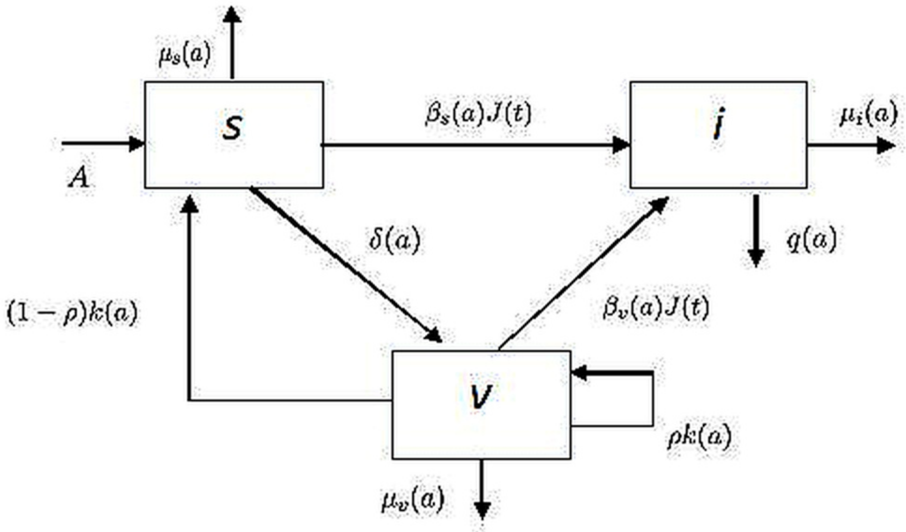

Figure 1: Diagram flux of the system (1).

and $\mu_i(\cdot)$ are the age-dependent per capita death for susceptible, infected, protected populations, respectively, with age a. The parameter ρ is the probability of returning again to the v-class, more precisely, it is the specific protection rate which highlights the concerned persons for the re-protection. The constant A represents the entering flux into the s-class. The functions $\delta(\cdot),k(\cdot)$ are, respectively, the protection rate, removing from v-class to s-class (rate of losing protection) and $\beta_s(\cdot),\beta_v(\cdot)$ are transmission rates. $q(.)$ is the recovering rate after spending a time in i-class, $\theta(.)$ represents the infectivity rate for an arbitrary infected person. See Figure 1.

We organize this research in the following form : After presenting the assumptions about the model data in the Preliminaries section, we will provide Volterra formulation for problem (1).We ensure the existence of a global compact attractor in section 4. Section 5 will offer the system of total trajectories that will enable us to study the global stability of solutions. Subsection 6.1 is dedicated to demonstrating the global stability of the trivial equilibrium (which always exists) in the case of $\mathcal{R}_0 \leq 1$. In subsection 6.2, we will discuss the emergence of the positive (endemic) equilibrium and the persistence of the disease, where $\mathcal{R}_0 > 1$, as well as its global stability through an appropriate Lyapunov function. Numerical simulation plays a crucial role in validating and illustrating theoretical findings in such studies. For example, in a recently published paper Benchaira et al., 2024, the authors in this paper show, by simulation, that the newly proposed estimator behaves well both in terms of bias and mean squared error. Similarly, in our work, theoretical results will be confirmed through numerical simulation examples, highlighting the interplay between theory and computation in epidemiological research. **Note that the theorems and proofs stated in Hathout et al., 2022 will not be reiterated here**.

2. Preliminaries

We assume that:

($\mathbf{H_1}$) All the parameters mentioned in model (1) are assumed to be positive, we also assume that $\mu_s, \mu_v, \mu_i, q, \theta \in L^\infty_+(\mathbb{R}^+) \setminus \{0_{L^\infty}\}$. In addition

$$M := \min\left\{ess \inf_{a\in\mathbb{R}_+}\{\mu_s(a)\}, ess \inf_{a\in\mathbb{R}_+}\{\mu_v(a)\}, ess \inf_{a\in\mathbb{R}_+}\{\mu_i(a)\}\right\} > 0.$$

($\mathbf{H_2}$) δ, k, β_s and β_v are positive, Lipschitz continuous functions in $\mathbb{R}^+$, with $\beta_s, \beta_v \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$.

In addition, we set the functional space for system (1)

$$X_+ := L^1_+\left(\mathbb{R}^+\right) \times L^1_+\left(\mathbb{R}^+\right) \times L^1_+\left(\mathbb{R}^+\right),$$

which is the positive cone of

$$X := L^1\left(\mathbb{R}^+\right) \times L^1\left(\mathbb{R}^+\right) \times L^1\left(\mathbb{R}^+\right),$$

equipped with the norm

$$\left\|\left(s(t,.), v(t,.), i(t,.)\right)\right\|_X = \int_0^\infty |s\left(t,a\right)| \, da + \int_0^\infty |v\left(t,a\right)| \, da + \int_0^\infty |i\left(t,a\right)| \, da.$$

The following theorem guarantees existence and uniqueness of solutions for (1):

Theorem 2.1. *Let $x_0 = (\bar{s}(.), \bar{v}(.), \bar{i}(.)) \in X_+$, then there exists a unique nonnegative solution $(s(.), v(.), i(.)) \in C(\mathbb{R}^+, L^1(\mathbb{R}^+)) \times C(\mathbb{R}^+, L^1(\mathbb{R}^+)) \times C(\mathbb{R}^+, L^1(\mathbb{R}^+))$ to system (1).*

Proof. By applying the Banach fixed point method we can demonstrate existence and uniqueness of the non-negative solution to (1) for any positive initial condition. This procedure is used in Bentout and Touaoula, 2015 and could be applied here. □

3. Volterra integral equation

By the characteristics method Webb, 1985, the PDE's system (1) can be expressed by Volterra equation as the following:

$$s(t,a) = \begin{cases} s(t-a,0)\pi_s(a)\Gamma_s(t-a,a), & t > a \geq 0, \\ \bar{s}(a-t)\frac{\pi_s(a)\Gamma_s(t-a,a)}{\pi_s(a-t)\Gamma_s(t-a,a-t)}, & a \geq t \geq 0, \end{cases}$$

with

$$\Gamma_s(t,a) = exp\left\{-\int_0^a \beta_s(\sigma)J(t+\sigma)d\sigma\right\}, \qquad \pi_s(a) = exp\left\{-\int_0^a \left(\mu_s(\sigma) + \delta(\sigma)\right) d\sigma\right\},$$

$$v(t,a) = \begin{cases} v(t-a,0)\pi_v(a)\Gamma_v(t-a,a), & t > a \geq 0, \\ \bar{v}(a-t)\frac{\pi_v(a)\Gamma_v(t-a,a)}{\pi_v(a-t)\Gamma_v(t-a,a-t)}, & a \geq t \geq 0, \end{cases}$$

where

$$\Gamma_v(t,a) = exp\left\{-\int_0^a \beta_v(\sigma)J(t+\sigma)d\sigma\right\}, \qquad \pi_v(a) = exp\left\{-\int_0^a (\mu_v(\sigma) + k(\sigma))d\sigma\right\}.$$

We also have

$$i(t,a)=\begin{cases} i(t-a,0)\pi_i(a), & t>a\geq 0,\\ \bar{i}(a-t)\frac{\pi_i(a)}{\pi_i(a-t)}, & a\geq t\geq 0,\end{cases}$$

with

$$\pi_i(a)=exp\left\{-\int_0^a(\mu_i(\sigma)+q(\sigma))d\sigma\right\}.$$

4. Global compact attractor

In the subsequent we set

$$\mathbb{E}_M=\left\{(s,v,i)\in X_+:\int_0^\infty s\left(t,a\right)da+\int_0^\infty v\left(t,a\right)da+\int_0^\infty i\left(t,a\right)da\leq\frac{A}{M}\right\}.$$

It is not difficult to show that λ is positively invariant. We can also prove that there exists a continuous semi-flow $\Phi(t,x_0)=\Phi_t(x_0)$ such that $\Phi_t(x_0)=(s(t,.),v(t,.),i(t,.))$ with $t\in\mathbb{R}^+, x_0\in X_+$, where (s,v,i) is solution of (1). Extracting some properties on the set $\mathbb{E}_M$ and the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ as well by the following proposition:

Proposition 4.1. *Hathout et al., 2022 Let Φ_t be the semi-flow of the system (1), then we have the following aspects*

(i) $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ *is point dissipative. Further, $\mathbb{E}_M$ attracts all point in X_+*

(ii) $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}\in\mathbb{E}_M$ *for all $t\geqslant 0$ and $x_0\in\mathbb{E}_M$.*

Theorem 4.2. *Hathout et al., 2022 The semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ engendered by system (1) is asymptotically smoothMagal and Thieme, 2004; Magal and Zhao, 2005. In addition, $\Phi_t(x_0)$ has a compact attractor $\boldsymbol{B}$ restrained to X_+. Moreover $\boldsymbol{B}$ attracts all bounded sets of X_+.*

5. Total trajectories

A total trajectory is a function ϕ that satisfies $\phi(t+r)=\Phi(t,\phi(r))$ for all $t\in\mathbb{R}$ and $r\geq 0$. Thus, for $\phi(t)=(s(t,.),v(t,.),i(t,.))$, $t\in\mathbb{R}$ and $a\geq 0$, we define a total trajectory as

$$\begin{cases} s(t,a)=s(t-a,0)\pi_s(a)\Gamma_s(t-a,a),\\ \Gamma_s(t,a)=exp\left\{-\int_0^a\beta_s(\sigma)J(t+\sigma)d\sigma\right\},\\ v(t,a)=v(t-a,0)\pi_v(a)\Gamma_v(t-a,a),\\ \Gamma_v(t,a)=exp\left\{-\int_0^a\beta_v(\sigma)J(t+\sigma)d\sigma\right\},\\ i(t,a)=i(t-a,0)\pi_i(a),\\ J(t)=\int_0^\infty\theta(a)i(t,a)da,\end{cases}$$

where $s(t,0)$, $v(t,0)$ and $i(t,0)$ are defined in (1).

Lemma 5.1. *Hathout et al., 2022 For all $x_0:=(\bar{s}(.),\bar{v}(.),\bar{i}(.))\in\boldsymbol{B}$, the following estimates hold true.*

$$\int_0^\infty s(t,a)da+\int_0^\infty v(t,a)da+\int_0^\infty i(t,a)da\leq\frac{A}{M}.$$

$$J(t)\leq||\theta||_\infty\frac{A}{M},$$

for all $t\in\mathbb{R}$ and there exist positive constants c_1 and c_2 such that

$$s(t,a)\geq c_1\pi_s(a),$$
$$v(t,a)\geq c_2\pi_v(a),$$

for all $t\in\mathbb{R}$ and $a\geq 0$.

6. Equilibria

6.1. Disease free equilibrium

In this section we prove that model (1) has always the trivial equilibrium which coincides with that of Hathout et al., 2022. Then, we can use the same arguments to prove the existence as well as the global stability of this state:

Theorem 6.1. *The disease free equilibrium is defined by* $E_0 = (s_0(a), p_0(a), 0)$, *where*

$$\begin{cases} s_0(a) = s_0(0)\pi_s(a), \\ p_0(a) = p_0(0)\pi_v(a), \qquad a > 0, \end{cases}$$

$$\text{with} \qquad \begin{cases} s_0(0) = A + (1-\rho)p_0(0)\int_0^\infty k(a)\pi_v(a)da, \\ \\ p_0(0) = \dfrac{A\int_0^\infty \delta(a)\pi_s(a)da}{1 - \int_0^\infty k(a)\pi_v(a)da((1-\rho)\int_0^\infty \delta(a)\pi_s(a)da + \rho)}. \end{cases}$$

Hence, we can define the basic reproduction rate $\mathcal{R}_0$ for model (1) by:

$$\mathcal{R}_0 = \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty (\beta_s(a)s_0(a) + \beta_v(a)v_0(a))\, da.$$

Remark that

$$\mathcal{R}_0 = \tilde{\mathcal{R}}_0 + \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty \beta_v(a)v_0(a)da.$$

where

$$\tilde{\mathcal{R}}_0 = \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty \beta_s(a)s_0(a)da$$

is the basic reproduction rate of model 1.1 in Hathout et al., 2022 and so,

$$\mathcal{R}_0 > \tilde{\mathcal{R}}_0$$

In a model of imperfect vaccination, vaccinated individuals can still become infected, allowing the infection to spread among them. This increases the proportion of the susceptible population, leading to a higher $\mathcal{R}_0$ compared to a perfect vaccination model, where vaccinated individuals are fully protected. While vaccination reduces the likelihood of infection, the persistence of susceptibility within the vaccinated population limits its control effect. Therefore, higher vaccination rates are required to control the infection in the case of imperfect vaccination.

Theorem 6.2. *Assume that* $\mathcal{R}_0 \leq 1$. *The disease free equilibrium* E_0 *is globally stable in* X_+.

6.2. Endemic equilibrium

The main objective of this section is to show the existence and global stability of the endemic equilibrium in the case where $\mathcal{R}_0 > 1$.

6.2.1 Existence

In this subsection our focus is on analyzing the existence of positive endemic equilibrium for model (1). This state verifies the following system:

$$\begin{cases} \dfrac{ds^*(a)}{da} = -(\mu_s(a)+\delta(a))s^*(a) - \beta_s(a)s^*(a)J^*, \\ \dfrac{dv^*(a)}{da} = -(\mu_v(a)+k(a))v^*(a) - \beta_v(a)v^*(a)J^*, \qquad a>0, \\ \dfrac{di^*(a)}{da} = -(\mu_i(a)+q(a))i^*(a), \qquad a>0, \\ s^*(0) = A + (1-\rho)\displaystyle\int_0^\infty k(a)v^*(a)da, \\ v^*(0) = \displaystyle\int_0^\infty \delta(a)s^*(a)da + \rho\int_0^\infty k(a)v^*(a)da, \\ i^*(0) = J^*\displaystyle\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da, \\ J^* = \displaystyle\int_0^\infty \theta(a)i^*(a)da, \end{cases} \tag{2}$$

which has the solution

$$\begin{cases} s^*(a) = s^*(0)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}, \\ v^*(a) = v^*(0)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}, \\ i^*(a) = i^*(0)\pi_i(a), \qquad a>0. \end{cases} \tag{3}$$

Theorem 6.3. *If $\mathcal{R}_0 > 1$, there exists the unique positive equilibrium denoted $E^* = (s^*(a), v^*(a), i^*(a))$.*

Proof. Firstly, using the equations of (2) and (3) we have

$$\begin{aligned} v^*(0) &= \int_0^\infty \delta(a)s^*(a)da + \rho\int_0^\infty k(a)v^*(a)da \\ &= s^*(0)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}da + \rho v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}da \\ &= \left(A + (1-\rho)v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}da\right)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}da \\ &+ \rho v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}da, \end{aligned}$$

and thus

$$v^*(0) = \frac{A\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}da}{1-\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}da\left((1-\rho)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}da+\rho\right)} \tag{4}$$

Next, suppose that $i^*(0) > 0$. Using the expression of J^* in (2) and dividing the following equation $i^*(0) = J^*\displaystyle\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da$ by $i^*(0)$ we obtain

$$1 = \int_0^\infty \theta(a)\pi_i(a)da\int_0^\infty \left(\beta_s(a)s^*(0)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma} + \beta_v(a)v^*(0)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}\right)da.$$

By employing the expression of $s^*(0)$ in (2), the last equation becomes:

$$1 = \tilde{\theta}\int_0^\infty \left(\beta_s(a)\left(A+(1-\rho)\tilde{k}v^*(0)\right)\pi_s(a)e^{-i^*(0)\tilde{\theta}\tilde{\beta}_s(a)} + \beta_v(a)v^*(0)\pi_v(a)e^{-i^*(0)\tilde{\theta}\tilde{\beta}_v(a)}\right)da, \tag{5}$$

where $\tilde{k} = \displaystyle\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}da$; $\tilde{\theta} = \displaystyle\int_0^\infty \theta(a)\pi_i(a)da$ and $\tilde{\beta}_{s,v}(a) = \displaystyle\int_0^a \beta_{s,v}(\sigma)d\sigma$.

Now, using the expression of $v^*(0)$ in (4) and the fact that $J^* = i^*(0)\tilde{\theta}$, we can rewrite problem (5) as the following :

$$1 = F(i^*(0)),$$

We can easily prove that F is a decreasing function. Furthermore, observe that $F(0) = \mathcal{R}_0$ and $\lim\limits_{y\to+\infty} F(y) = 0$. Therefore, problem (5) has a unique positive solution if $\mathcal{R}_0 > 1$. The proof is reached. □

6.2.2 Main results of uniform persistence

For the purpose of the well posedness of the Lyapunov function obtained in the next section we will show the persistence result. So, we define the following sets:

$$X^0 = \left\{ (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot)) \in X_+; \quad \int_0^\infty \theta(a)\bar{i}(a)da > 0 \right\}$$

$$\partial X^0 = \left\{ (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot)) \in X_+; \quad \int_0^\infty \theta(a)\bar{i}(a)da = 0 \right\}.$$

So, we write $X_+ = X^0 \cup \partial X^0$. For $x_0 = (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot))$, we also denote

$$M_\partial = \{x_0 \in \partial X^0; \ \ \Phi_t(x_0) \in \partial X^0, \ \ \text{for all } t \geq 0\}.$$

We have the following theorems:

Lemma 6.4. *The subset X^0 is positively invariant under the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$. Furthermore, the disease free equilibrium is globally asymptotically stable for the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ restricted to M_∂.*

Theorem 6.5. *Smith and Zhao, 2001 Assume that $\mathcal{R}_0 > 1$, the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ is uniformly persistent with respect to $(X^0, \partial X^0)$, i.e., there exists $\epsilon > 0$ which is independent of initial values such that $\liminf_{t\to\infty} \int_0^\infty \theta(a)i(t,a)da \geq \epsilon$ for all $x_0 \in X^0$. Moreover, there exists a compact subset $\boldsymbol{B}_0$ of X^0 which is a global attractor for $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ in X^0.*

Lemma 6.6. *For all $x_0 \in \boldsymbol{B}_0, a > 0$ and $t \in \mathbb{R}$, there exist positive constant c such that:*

$$\frac{s(t,a)}{s^*(a)} > c, \qquad \frac{v(t,a)}{v^*(a)} > c, \qquad \frac{i(t,a)}{i^*(a)} > c.$$

6.2.3 Global stability

Theorem 6.7. *Assume that $\mathcal{R}_0 > 1$. The endemic equilibrium is globally stable in $\boldsymbol{B}_0 \subset X^0$.*

Proof. We define $H(x) = x - ln(x) - 1$ and consider the following Lyapunov functional:

$$W(t) = \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s(a)da + \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v(a)da + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i(a)da$$

with $\quad \phi_s(a) = \dfrac{s^*(a)}{i^*(0)}, \qquad \phi_v(a) = \dfrac{v^*(a)}{i^*(0)}, \qquad \phi_i(a) = \dfrac{\int_a^\infty \theta(s)i^*(s)ds}{\int_0^\infty \theta(a)i^*(a)da}, \qquad a \geq 0$

Note that the functions ϕ_s, ϕ_v and ϕ_i verify the following problems:

$$\begin{cases} \phi_s'(a) = -(\mu_s(a) + \delta(a) + \beta_s(a)J^*)\dfrac{s^*(a)}{i^*(0)}, \\ \phi_s(0) = \dfrac{s^*(0)}{i^*(0)}, \end{cases} \qquad \begin{cases} \phi_v'(a) = -(\mu_v(a) + k(a) + \beta_v(a)J^*)\dfrac{v^*(a)}{i^*(0)} \\ \phi_v(0) = \dfrac{v^*(0)}{i^*(0)}, \end{cases}$$

$$\begin{cases} \phi_i'(a) = -\dfrac{\theta(a)i^*(a)}{\int_0^\infty \theta(a)i^*(a)da}, \\ \phi_i(0) = 1 \end{cases}$$

Set

$$W_s(t) := \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s(a)da, \quad W_v(t) := \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v(a)da,$$

$$W_i(t) := \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i(a)da.$$

Using Lemma 3.3 in Hathout et al., 2022, we obtain

$$\begin{aligned} W_s'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da - J(t)\int_0^\infty \beta_s(a)\phi_s(a)\frac{s(t,a)}{s^*(a)}H'\left(\frac{s(t,a)}{s^*(a)}\right)da \\ &+ J^*\int_0^\infty \beta_s(a)\phi_s(a)\frac{s(t,a)}{s^*(a)}H'\left(\frac{s(t,a)}{s^*(a)}\right)da \end{aligned}$$

$$\begin{aligned} W_v'(t) &= \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) + \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da - J(t)\int_0^\infty \beta_v(a)\phi_v(a)\frac{v(t,a)}{v^*(a)}H'\left(\frac{v(t,a)}{v^*(a)}\right)da \\ &+ J^*\int_0^\infty \beta_v(a)\phi_v(a)\frac{v(t,a)}{v^*(a)}H'\left(\frac{v(t,a)}{v^*(a)}\right)da \end{aligned}$$

$$W_i'(t) = \phi_i(0)H\left(\frac{i(t,0)}{i^*(0)}\right) + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i'(a)da$$

Using the fact that $H'(x) = 1 - \dfrac{1}{x}$, we get

$$\begin{aligned} W_s'(t) + W_v'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da + \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) \\ &+ J^*\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da \\ &- J^*\int_0^\infty (\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a))\,da + J(t)\int_0^\infty (\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a))\,da \\ &- J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da, \\ &+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da \end{aligned}$$

In addition, we have

$$J(t)\int_0^\infty (\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a))\,da = \frac{J(t)}{J^*}\frac{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} = \frac{J(t)}{J^*}.$$

$$J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da = \frac{J(t)}{J^*}\frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da},$$

$$J^*\int_0^\infty (\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a))\,da = \frac{J^*}{J^*}\frac{\int_0^\infty \beta_s(a)s^*(a)da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} = 1,$$

$$J^*\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da = \frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da}.$$

Since $W' = W_s' + W_v' + W_i'$ and

$$\begin{aligned} H\left(\frac{i(t,0)}{i^*(0)}\right) &= H\left(\frac{J(t)}{J^*}\frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da}\right) \\ &+ \frac{J(t)}{J^*}\frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty \beta_s(a)s^*(a)da} \\ &- \ln\frac{J(t)}{J^*} - \ln\frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty \beta_s(a)s^*(a)da} - 1, \end{aligned}$$

it follows that

$$\begin{aligned} W'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da + \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) \\ &+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i'(a)da \\ &+ H\left(\frac{\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da}\right) + H\left(\frac{J(t)}{J^*}\right) \end{aligned}$$

On the other hand,

$$H\left(\frac{s(t,0)}{s^*(0)}\right) = H\left(\frac{A}{s^*(0)}\cdot 1 + \frac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)}\frac{\int_0^\infty k(a)v(t,a)da}{\int_0^\infty k(a)v^*(a)da}\right).$$

Since H is convex and $\dfrac{A}{s^*(0)}+\dfrac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)}=1$ then,

$$H\left(\frac{s(t,0)}{s^*(0)}\right)\le \frac{A}{s^*(0)}\underbrace{H(1)}_{=0}+\frac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)}H\left(\frac{\int_0^\infty k(a)v^*(a)\frac{v(t,a)}{v^*(a)}da}{\int_0^\infty k(a)v^*(a)da}\right).$$

By Jensen inequality, this last inequality leads to

$$H\left(\frac{s(t,0)}{s^*(0)}\right)\le \frac{(1-\rho)}{s^*(0)}\int_0^\infty k(a)v^*(a)H\left(\frac{v(t,a)}{v^*(a)}\right)da.$$

$$\begin{aligned}
H\left(\frac{v(t,0)}{v^*(0)}\right) &= H\left(\frac{1}{v^*(0)}\left(\int_0^\infty \delta(a)s(t,a)da+\rho\int_0^\infty k(a)v(t,a)da\right)\right),\\
&= H\left(\frac{\int_0^\infty \delta(a)s^*(a)da}{v^*(0)}\frac{\int_0^\infty \delta(a)s(t,a)da}{\int_0^\infty \delta(a)s^*(a)da}+\rho\frac{\int_0^\infty k(a)v^*(a)da}{v^*(0)}\frac{\int_0^\infty k(a)v(t,a)da}{\int_0^\infty k(a)v^*(a)da}\right),\\
&\le \frac{\int_0^\infty \delta(a)s^*(a)da}{v^*(0)}H\left(\frac{\int_0^\infty \delta(a)s^*(a)\frac{s(t,a)}{s^*(a)}da}{\int_0^\infty \delta(a)v^*(a)da}\right)\\
&+ \rho\frac{\int_0^\infty k(a)v^*(a)da}{v^*(0)}H\left(\frac{\int_0^\infty k(a)v^*(a)\frac{v(t,a)}{v^*(a)}da}{\int_0^\infty k(a)v^*(a)da}\right),\\
&\le \frac{1}{v^*(0)}\int_0^\infty \delta(a)s^*(a)H\left(\frac{s(t,a)}{s^*(a)}\right)da+\frac{\rho}{v^*(0)}\int_0^\infty k(a)v^*(a)H\left(\frac{v(t,a)}{v^*(a)}\right)da.
\end{aligned}$$

$$\begin{aligned}
H\left(\frac{J(t)}{J^*}\right)=H\left(\frac{\int_0^\infty \theta(a)i(t,a)da}{\int_0^\infty \theta(a)i^*(a)da}\right) &= H\left(\frac{\int_0^\infty \theta(a)i^*(a)\frac{i(t,a)}{i^*(a)}da}{\int_0^\infty \theta(a)i^*(a)da}\right)\\
&\le \frac{\int_0^\infty \theta(a)i^*(a)H\left(\frac{i(t,a)}{i^*(a)}\right)da}{\int_0^\infty \theta(a)i^*(a)da}.
\end{aligned}$$

$$\begin{aligned}
H\left(\frac{\int_0^\infty(\beta_s(a)s(t,a)+\beta_v(a)v(t,a))\,da}{\int_0^\infty(\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}\right) &\le \frac{\int_0^\infty \beta_s(a)s^*(a)H\left(\frac{s(t,a)}{s^*(a)}\right)da}{\int_0^\infty(\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}\\
&+ \frac{\int_0^\infty \beta_v(a)v^*(a)H\left(\frac{v(t,a)}{v^*(a)}\right)da}{\int_0^\infty(\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
W'(t) &\le \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\left(\phi_s'(a)+\frac{\beta_s(a)s^*(a)}{\int_0^\infty(\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}+\frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a)\right)da\\
&+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\left(\phi_v'(a)+\frac{\beta_v(a)v^*(a)}{\int_0^\infty(\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}+\tau k(a)v^*(a)\right)da\\
&+ \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\underbrace{\left(\phi_i'(a)+\frac{\theta(a)i^*(a)}{\int_0^\infty \theta(a)i^*(a)da}\right)}_{=0}da.
\end{aligned}$$

where

$$\tau = \rho \frac{\phi_v(0)}{v^*(0)} + (1-\rho)\frac{\phi_s(0)}{s^*(0)} = \rho \frac{1}{v^*(0)}\frac{v^*(0)}{i^*(0)} + (1-\rho)\frac{1}{i^*(0)} = \frac{1}{i^*(0)},$$

Let

$$L_s(a) := \phi_s'(a) + \frac{\beta_s(a)s^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a)$$

Replacing ϕ_s' by its expression we get:

$$\begin{aligned} L(a) &= -\frac{s^*(a)}{i^*(0)}(\mu_s(a) + \delta(a) + \beta_s(a)J^*) + \frac{\beta_s(a)s^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a), \\ &= -\frac{1}{i^*(0)}\mu_s(a)s^*(a) + \delta(a)s^*(a)\left(\frac{\phi_v(0)}{v^*(0)} - \frac{1}{i^*(0)}\right) \\ &+ \beta_s(a)s^*(a)\left(\frac{1}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} - \frac{J^*}{i^*(0)}\right), \end{aligned}$$

employing the equations of $i^*(0)$ and $\phi_v(0)$ we obtain

$$\begin{cases} \dfrac{1}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} - \dfrac{J^*}{i^*(0)} = 0, \\ \dfrac{\phi_v(0)}{v^*(0)} - \dfrac{1}{i^*(0)} = \dfrac{1}{v^*(0)}\dfrac{v^*(0)}{i^*(0)} - \dfrac{1}{i^*(0)} = 0, \end{cases}$$

then

$$L_s(a) = -\frac{1}{i^*(0)}\mu_s(a)s^*(a)$$

Similarly, we prove that :

$$\begin{aligned} L_v(a) &= \phi_v'(a) + \frac{\beta_v(a)v^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{k(a)v^*(a)}{i^*(0)}, \\ &= -\mu_v(a)\frac{v^*(a)}{i^*(0)}. \end{aligned}$$

Finally, the derivative W' verifies the following inequality

$$\begin{aligned} W'(t) &\leq -\frac{1}{i^*(0)}\int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\mu_s(a)s^*(a)da - \frac{1}{i^*(0)}\int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\mu_v(a)v^*(a)da, \\ &\leq 0. \end{aligned}$$

We know that $\frac{d}{dt}W(t) = 0$ implies that $s(t,a) = s^*(a)$ and $v(t,a) = v^*(a)$ for all $t \in \mathbb{R}$ and $a \geq 0$. We replace these into the first equation of (1), we conclude that $J(t) = J^*$ and so $i(t,0) = i^*(0)$. hence, it follows that $i(t,a) = i^*(a)$ for all $t \in \mathbb{R}$ and $a \geq 0$. Therefore, the largest invariant set with the property $\frac{d}{dt}W(t) = 0$ is $\{(s^*(a), v^*(a), i^*(a))\}$. Finally, by employing the same argument as in the proof of Theorem 4.1 in **ST** we reach the result. □

7. Discussion

In this study, we investigated the model of imperfect vaccination, obtaining results based on the values of the basic reproduction number, $\mathcal{R}_0$. Specifically, if $\mathcal{R}_0 \leq 1$, we observe the extinction of the disease, as expressed by the stability of the unique equilibrium (trivial equilibrium). Conversely, when $\mathcal{R}_0 > 1$, the disease persists in the population, as indicated by the stability of the second equilibrium (positive equilibrium).

By considering the same set of parameters as in Hathout et al., 2022, the numerical results insure the threshold dynamics obtained in the theoretical part. Indeed, in Fig.2 we remark that the extinction scenario of the infection holds, wherein this case we obtained that $\mathcal{R}_0 = 2.7222.10^{-8} < 1$, which confirms Theorem 6.2. Besides, in Fig.3, we remark the persistence of infection to a positive value, where it is obtained that $\mathcal{R}_0 = 7.0118 > 1$. This figure ensures the main result of Theorem 6.7.

The issues related to imperfect vaccination are diverse and can pose challenges in the fight against infectious

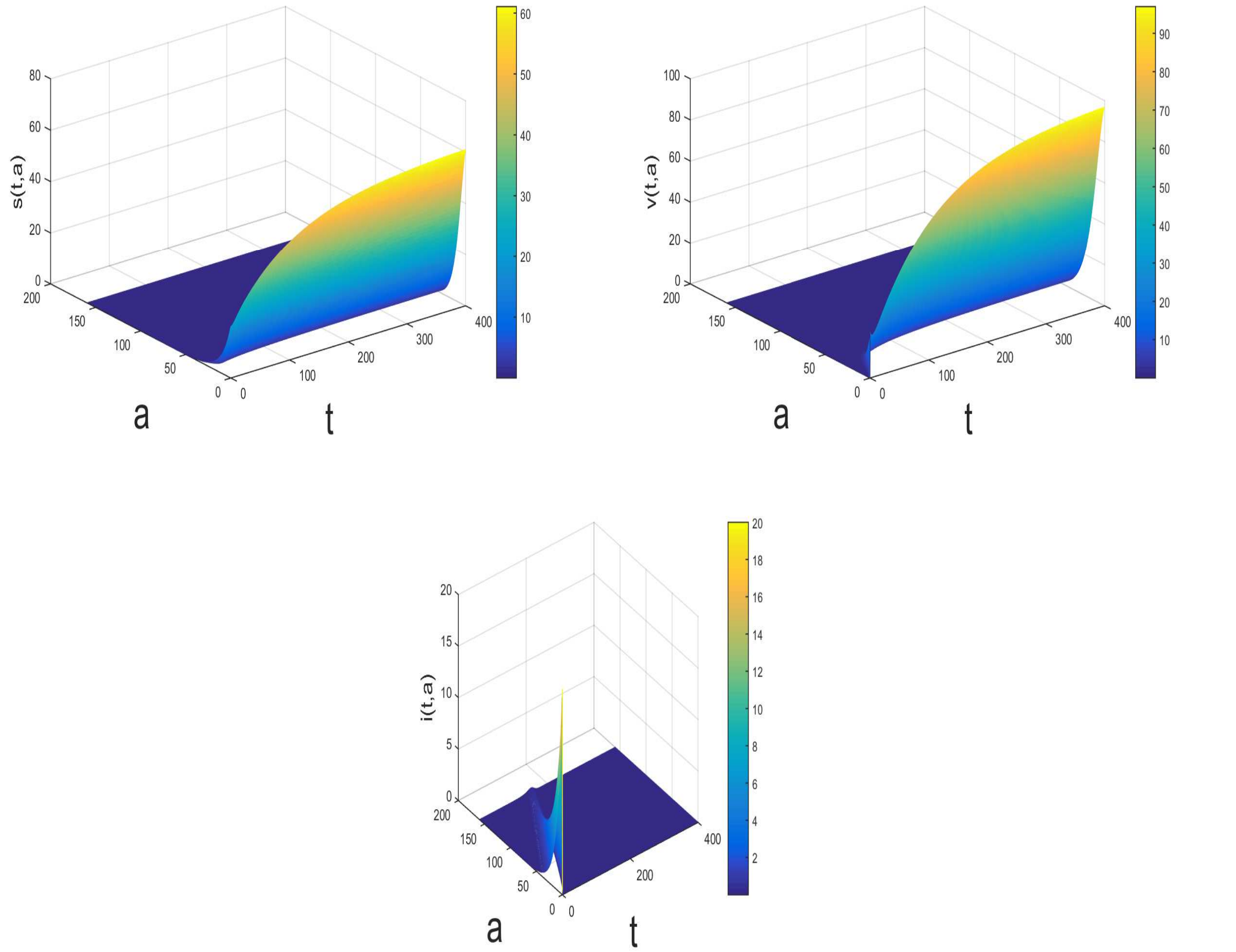

Figure 2: The global stability of the disease free equilibrium in the case of $\mathcal{R}_0 = 2.7222.10^{-8} < 1$.

diseases. Some of these problems include the possibility of developing partial immunity after vaccination, the limited duration of immunity, the variable effectiveness of vaccines, selective protection against certain serotypes of pathogens, rare risks of post-vaccination infection, viral adaptation, and variable immune responses in individuals.

Partial immunity may leave some individuals vulnerable to infection, although the severity of the disease may be reduced. Additionally, the waning immunity over time requires regular boosters to maintain adequate protection. The variable effectiveness of vaccines, influenced by factors such as age and general health, can lead to disparities in protection within the population.

Furthermore, the type of vaccination strategy plays a critical role in the success of vaccination efforts. Systematic vaccination campaigns, as seen with diseases like measles, often result in higher coverage and more consistent protection, leading to herd immunity. In contrast, non-systematic vaccination, such as for seasonal flu, may require continuous efforts and adaptation to address seasonal variation and emerging strains, often with lower overall effectiveness in the long run.

To overcome these problems, several solutions can be considered. Firstly, ongoing research to improve the duration of vaccine immunity and develop more durable formulations is crucial. Efforts to understand variable immune responses could allow for vaccine customization based on individual profiles.

Education and communication also play a crucial role. It is important to inform the public about the benefits of vaccination despite its imperfections, emphasizing that even partial protection can reduce the severity of diseases and contribute to prevention of spread.

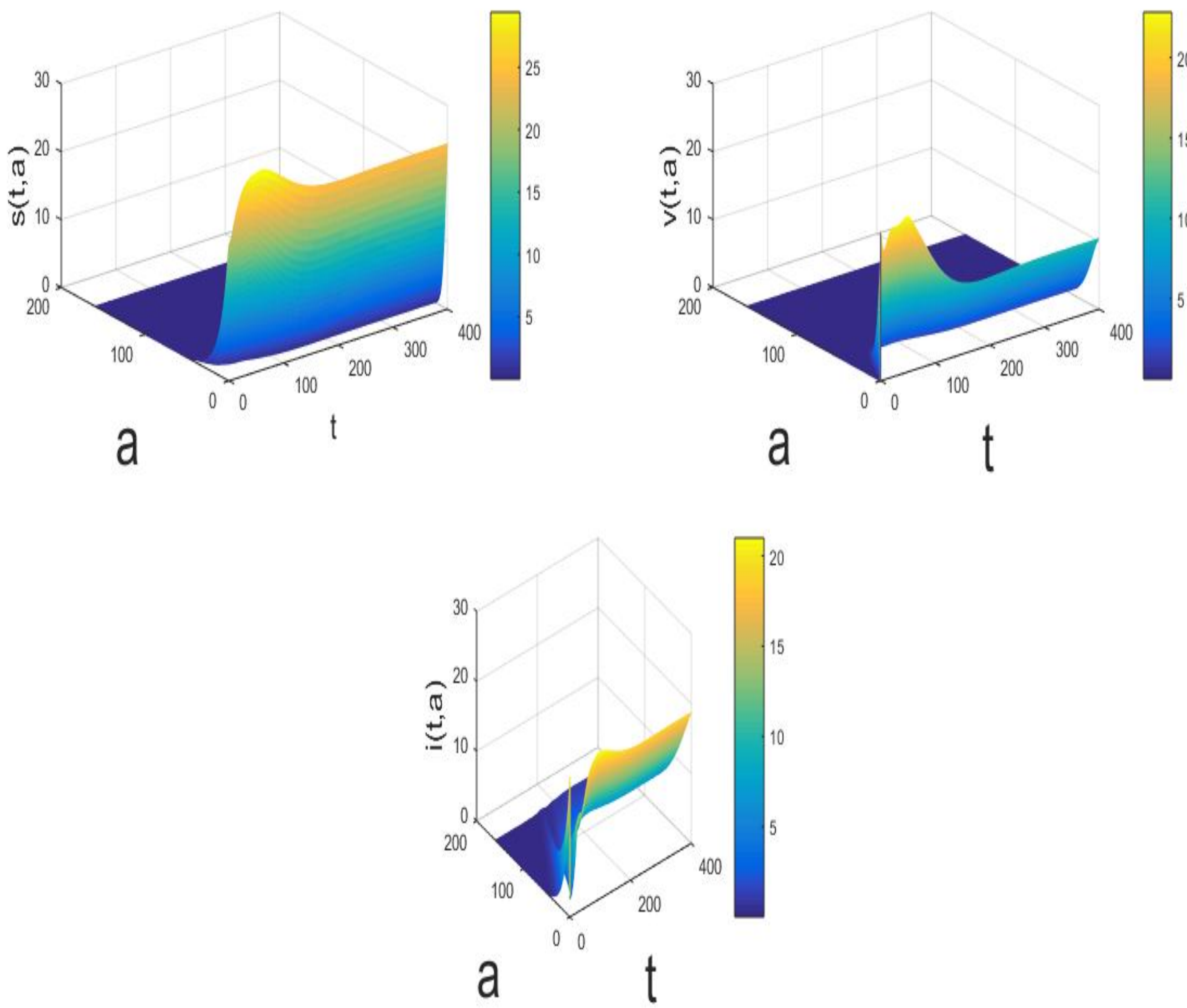

Figure 3: The global stability of the endemic equilibrium in the case of $\mathcal{R}_0 = 7.0118 > 1$.

Continuous surveillance of outbreaks and rapid detection of emerging viral strains are fundamental. This could lead to swift adjustments of vaccine formulations to maintain effective protection against new variants.

Ultimately, research, education, surveillance, and constant innovation are key elements in addressing issues associated with imperfect vaccination. By combining these approaches, it is possible to strengthen the fight against infectious diseases and improve the effectiveness of vaccination programs.

Acknowledgment

The author would like to thank to express their gratitude to the anonymous reviewers for their constructive feedback and the editor for their thorough evaluation of this manuscript. The author declares no conflicts of interest. The authors are partially supported by the DGRSDT of Algeria No.C00L03UN130120200004

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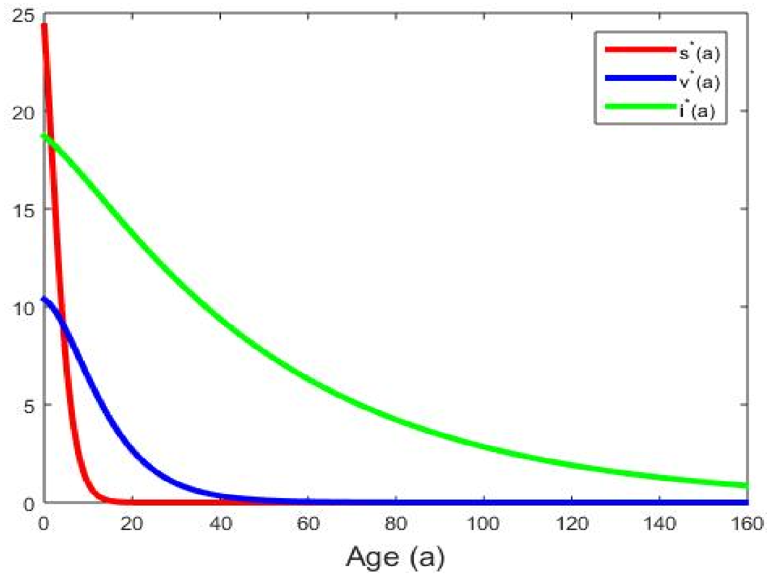

Figure 4: The value of the endemic equilibrium state obtained in Fig.3.

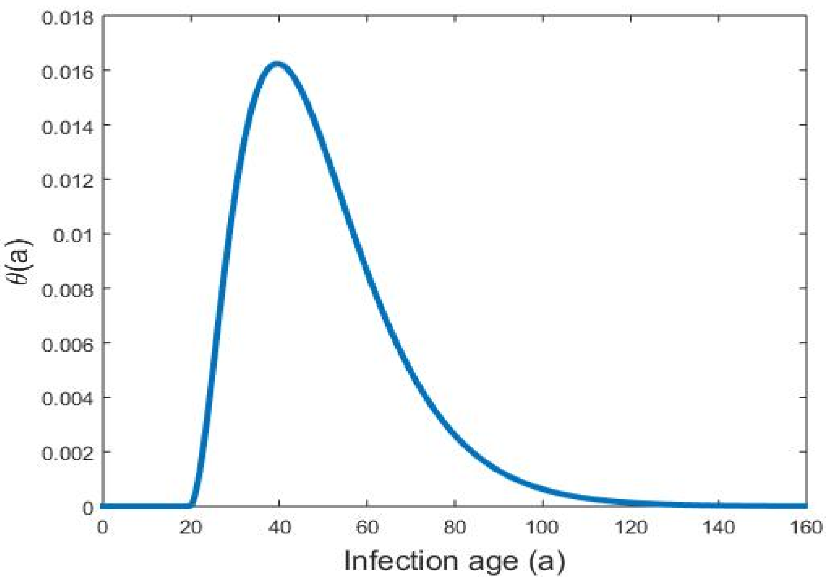

Figure 5: Graphical representation of the infectivity rate θ.

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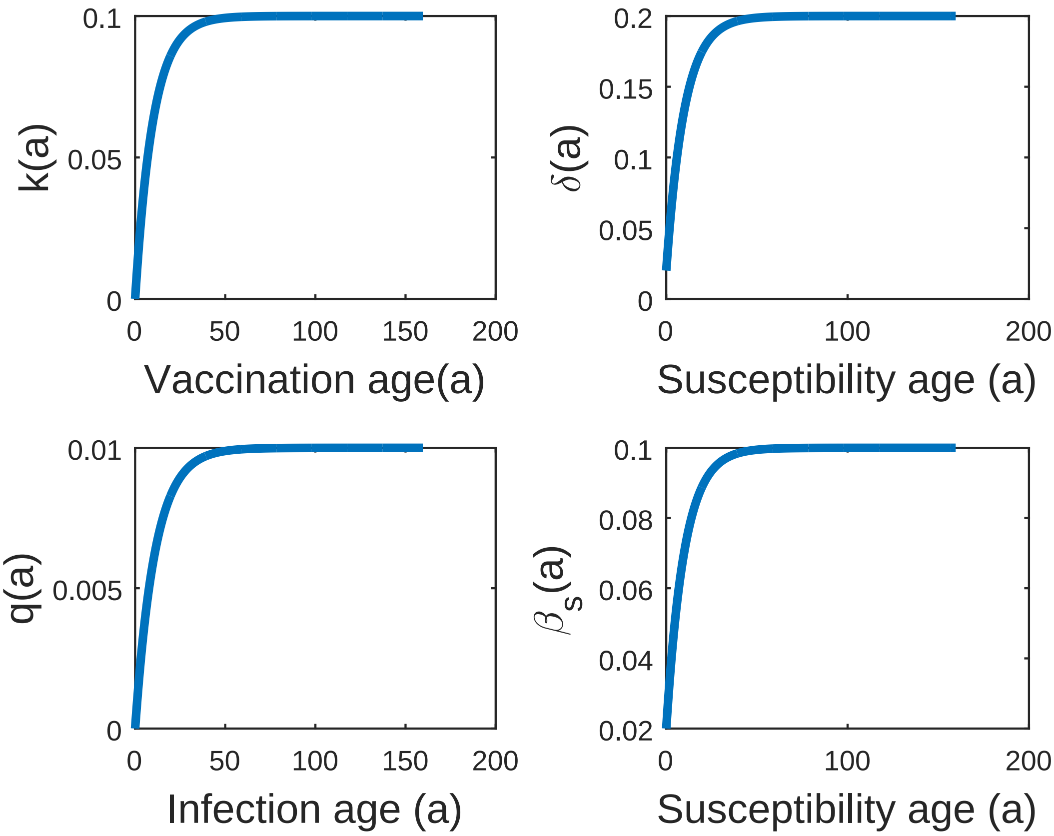

Figure 6: Graphical representation of the functions: k, δ, q and $\beta_s = \beta_v$.

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On Fixed Point Theorems for Self-Mappings in Complex Metric Spaces with Special Functions

IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

On Fixed Point Theorems for Self-Mappings in Complex Metric Spaces with Special Functions

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DOI: https://doi.org/10.69717/ijams.v2.i1.116

abstract

This paper delves into the forefront of fixed point theory, focusing on recent advancements within the context of contraction mappings in complex metric spaces. The study introduces a novel perspective by incorporating the pivotal role of control functions in elucidating the behavior and properties of fixed points. We investigate the interplay between contraction mappings and complex metric spaces using a control function. We provide an example to illustrate our findings.

keywords

Complex metric space, Special function, Generalized contraction.

2020 Mathematics Subject Classification

47H10 · 54H25 ·

1. Introduction and Preliminaries

In recent years, fixed point theory has witnessed a surge of interest and innovation, particularly in the exploration of contraction mappings within the intricate realm of complex metric spaces, see (Bhatt et al., 2011, Kang et al., 2013, Kutbi et al., 2013, Ahmad et al., 2013, Manro, 2013, Mohanta and Maitra, 2012, Rouzkard and Imdad, 2012, Sintunavarat and Kumam, 2012, Sitthikul and Saejung, 2012, Verma and Pathak, 2013). This paper aims to contribute to this evolving discourse by investigating novel perspectives and advancements in the field, with a particular focus on the pivotal role of control functions in shaping the dynamics of fixed point iterations. Firstly, in the preliminary table, we need to define a new partial order relation $\precsim$ on C.
Let C be the set of complex numbers and $z_1, z_2 \in C$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following cases is satisfied:

$$\begin{aligned} Re(z_1) = Re(z_2), Im(z_1) < Im(z_2), \\ Re(z_1) < Re(z_2), Im(z_1) = Im(z_2), \\ Re(z_1) < Re(z_2), Im(z_1) < Im(z_2), \\ Re(z_1) = Re(z_2), Im(z_1) = Im(z_2). \end{aligned}$$

we write $z_1 \precnsim z_2$ if $z_1 \precnsim z_2$ and $z_1 \neq z_2$,and we will write $z_1 \prec z_2$ if only (3) is satisfied.Note that

$$0 \precsim z1 \precnsim z2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2 \text{ and } z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$
$$0 \precsim z1 \precsim z2 \Rightarrow |z_1| \leq |z_2|.$$

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Communicated Editor: A. Chala
Manuscript received *Dec 23, 2023;* ***revised*** *Sep 16, 2024;* ***accepted*** *Dec 11, 2024;* ***published on line*** *May 12, 2025..*

Definition 1.1. Azam Azam et al., 2011 Let X be a nonempty set. Suppose that the function $d : X \times X \to \mathbb{C}$,satisfies.
(a) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$,
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(c) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
d is called a complex valued metric in X and The pair (X, d) is called a complex valued metric space.

Example 1.2. Sintunavarat and Kumam, 2012 Let $X = \mathbb{C}$ Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(z_1, z_2) = \exp(ik) |z_1 - z_2|^2,$$

where $k \in \left[0, \frac{\pi}{2}\right]$. Then (X, d) is a complex valued metric space.

Definition 1.3. Azam Azam et al., 2011 Suppose that (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $x \in X$., We find that
(i) the sequence $\{x_n\}$ converges to $x_0 \in X$ if for every $0 < c \in \mathbb{C}$, there exists an integer N such that $d(x_n, x_0) < c$ for all $n \geq N$.
we write $x_n \to x_0$.
(ii) the sequence $\{x_n\}$ is a Cauchy sequence if for every $0 < c \in \mathbb{C}$, there exists an integer N such that $d(x_n, x_m) < c$ for all $n, m \geq N$.
(iii) the metric space (X, d) is complete, if every Cauchy sequence in X converges to a point in X.

Lemma 1.4. *Azam Azam et al., 2011 Let (X, d) be a complex valued metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converge to x_0 if and only if $|d(x_n, x_0)| \to 0$ as $n \to \infty$.*

Lemma 1.5. *Azam Azam et al., 2011 Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$,*

Lemma 1.6. *Azam Azam et al., 2011 let $\{x_n\}$ be a sequence in X and $h \in [0, 1)$ .if $a_n = |d(x_n, x_{n+1})|$ satisfies*

$$a_n \leq h a_{n-1}, \text{ for all } n \in N,$$

then $\{x_n\}$ is a Cauchy sequence.

2. Main results

Firstly, in this chapter, we will need to utilize the following assumption.
Throughout this work, Let (X, d) be a complex valued metric space and let $S, T : X \to X$.

Proposition 2.1. *Let $x_0 \in X$ and defined the sequence $\{x_n\}$ be defined by*

$$x_{2n+1} = Sx_{2n}, \ x_{2n+2} = Tx_{2n+1}, \text{ for all } n = 0, 1, 2, ...$$

Assume that there exists a control function $\gamma : X \times X \to [0, 1)$ satisfying.

$$\gamma(TSx, y) \leq \gamma(x, y) \text{ and } \gamma(x, STy) \leq \gamma(x, y)$$

for all $x, y \in X$. then

$$\gamma(x_{2n}, y) \leq \gamma(x_0, y) \text{ and } \gamma(x, x_{2n+1}) \leq \gamma(x, x_1)$$

for all $x, y \in X$ and $n = 0, 1, 2, ...$

Proof. let $x, y \in X$ and $n = 0, 1, 2, ...$ then we have

$$\begin{aligned}\gamma(x_{2n}, y) &= \gamma(TSx_{2n-2}, y) \leq \gamma(x_{2n-2}, y) \\ &= \gamma(TSx_{2n-4}, y) \leq ... \leq \gamma(x_0, y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\gamma(x, x_{2n+1}) &= \gamma(x, STx_{2n-1}) \leq \gamma(x, x_{2n-1}) \\ &= \gamma(x, STx_{2n-3}) \leq ... \leq \gamma(x, x_1).\end{aligned}$$

□

Theorem 2.2. *Let (X, d) be a complex valued metric space and let $S, T : X \to X$. if there exist the control function $\gamma : X \times X \to [0, 1)$ such that for all $x, y \in X$:*

(a)

$$\gamma(TSx, y) \leq \gamma(x, y) \text{ and } \gamma(x, STy) \leq \gamma(x, y);$$

(b)

$$\gamma(x_0, x_1) < 1, \tag{1}$$

(c)

$$d(Sx, Ty) \precsim \gamma(x, y)\frac{d(x, Sx)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(y, x)}, \tag{2}$$

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ be defined by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$,$n = 0, 1, 2, ...$Now by (2), Then

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim \gamma(x_{2n}, x_{2n+1})\frac{d(x_{2n}, Sx_{2n})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n}) + d(x_{2n+1}, x_{2n})} \\ &\precsim \gamma(x_{2n}, x_{2n+1})\frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + d(x_{2n+1}, x_{2n})} \\ &\precsim \gamma(x_{2n}, x_{2n+1})\frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+1}, x_{2n+2})} \\ &\precsim \gamma(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1}), \end{aligned}$$

Taking the modulus, we get

$$|d(x_{2n+1}, x_{2n+2})| \leq \gamma(x_{2n}, x_{2n+1})|d(x_{2n}, x_{2n+1})|.$$

Now by Proposition 2.1, therefore

$$\begin{aligned} |d(x_{2n+1}, x_{2n+2})| &\leq \gamma(x_0, x_{2n+1})|d(x_{2n}, x_{2n+1})| \\ &\leq \gamma(x_0, x_1)|d(x_{2n}, x_{2n+1})| \end{aligned}$$

which yeilds that

$$|d(x_{2n+1}, x_{2n+2})| \leq \gamma(x_0, x_1)|d(x_{2n}, x_{2n+1})|.$$

Similarly, we get

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Tx_{2n+1}, Sx_{2n+2}) \\ &\precsim \gamma(x_{2n+2}, x_{2n+1})\frac{d(x_{2n+2}, Sx_{2n+2})d(x_{2n+1}, Tx_{2n+1})}{1 + d(x_{2n+2}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n+2}) + d(x_{2n+1}, x_{2n+2})} \\ &\precsim \gamma(x_{2n+2}, x_{2n+1})\frac{d(x_{2n+2}, x_{2n+3})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+2}) + d(x_{2n+1}, x_{2n+3}) + d(x_{2n+1}, x_{2n+2})} \\ &\precsim \gamma(x_{2n+2}, x_{2n+1})\frac{d(x_{2n+2}, x_{2n+3})d(x_{2n+1}, x_{2n+2})}{1 + d(x_{2n+2}, x_{2n+3})} \\ &\precsim \gamma(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}), \end{aligned}$$

Taking the modulus, we get

$$|d(x_{2n+2}, x_{2n+3})| \leq \gamma(x_{2n+2}, x_{2n+1})|d(x_{2n+2}, x_{2n+1})|.$$

Now by Proposition 2.1, therefore

$$\begin{aligned} |d(x_{2n+2}, x_{2n+3})| &\leq \gamma(x_0, x_{2n+1})|d(x_{2n+2}, x_{2n+1})| \\ &\leq \gamma(x_0, x_1)|d(x_{2n+2}, x_{2n+1})| \end{aligned}$$

which yeilds that

$$|d(x_{2n+2}, x_{2n+3})| \leq \gamma(x_0, x_1)|d(x_{2n+1}, x_{2n+2})|.$$

Since $a = \gamma(x_0, x_1) < 1$,
thus we have,

$$|d(x_{2n+2}, x_{2n+3})| \leq a|d(x_{2n+1}, x_{2n+2})|,$$

or in fact

$$|d(x_n, x_{n+1})| \leq a|d(x_{n-1}, x_n)| \text{ for all } n \in N.$$

From lemma 1.6,we have $\{x_n\}$ is a Cauchy sequence in (X, d).Since X is complete,there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Next we show that u is a fixed point of S.
Now by (2) and Proposition 2.1, we can write

$$\begin{aligned} d(u, Su) &\precsim d(u, Tx_{2n+1}) + d(Tx_{2n+1}, Su) \\ &= d(u, Tx_{2n+1}) + d(Su, Tx_{2n+1}) \\ &\precsim d(u, Tx_{2n+1}) + \gamma(u, x_{2n+1}) \frac{d(u, Su)d(x_{2n+1}, Tx_{2n+1})}{1 + d(u, Tx_{2n+1}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)} \\ &\precsim d(u, x_{2n+2}) + \gamma(u, x_1) \frac{d(u, Su)d(x_{2n+1}, x_{2n+2})}{1 + d(u, x_{2n+2}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)}. \end{aligned}$$

on Making $n \to \infty$,reduces by making the modulus, we get

$$\begin{aligned} |d(u, Su)| &\leq \mu(u, x_1)|d(u, Su)| \\ &\leq (\gamma(u, x_1))|d(u, Su)| \\ &< |d(u, Su)|, \end{aligned}$$

which is contradiction. So, $Su = u$. Similarly, One can prove that u is a fixed point of T.by (2) and Proposition 2.1, we can write

$$\begin{aligned} d(u, Tu) &\precsim d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \\ &= d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\ &\precsim d(u, x_{2n+1}) + \gamma(x_{2n}, u) \frac{d(x_{2n}, Sx_{2n})d(u, Tu)}{1 + d(x_{2n}, Tu) + d(u, Sx_{2n}) + d(x_{2n}, u)} \\ &\precsim d(u, x_{2n+1}) + \gamma(x_0, u) \frac{d(x_{2n}, x_{2n+1})d(u, Tu)}{1 + d(x_{2n}, Tu) + d(u, x_{2n+1}) + d(x_{2n}, u)}. \end{aligned}$$

on Making $n \to \infty$,reduces to

$$d(u, Tu) \precsim \mu(x_0, u)d(u, Tu),$$

Taking the modulus, we get

$$\begin{aligned} |d(u, Tu)| &\leq \mu(x_0, u)|d(u, Tu)| \\ &\leq (\gamma(x_0, u))|d(u, Tu)| \\ &< |d(u, Tu)|, \end{aligned}$$

which is contradiction. So, $Tu = u$. We present to prove the uniqueness of the common fixed point of S and T. For this,Assume that the existence u^* is a second common fixed point. we have

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\precsim \gamma(u, u^*) \frac{d(u, Su)d(u^*, Tu^*)}{1 + d(u, Tu^*) + d(u^*, Su) + d(u, u^*)} \end{aligned}$$

which implies that

$$d(u, u^*) = 0.$$

Thus $u = u^*$, completing the proof of the theorem. □

Corollary 2.3. *Let (X, d) be a complex-valued metric space and let $S : X \to X$. If there exists control function $\gamma : X \times X \to [0, 1)$ such that for all $x, y \in X$ we have*

$$\gamma(S^2x, y) \leq \gamma(x, y) \text{ and } \gamma(x, S^2y) \leq \gamma(x, y);$$

$$\gamma(x, y) < 1;$$

$$d(Sx, Sy) \precsim \gamma(x, y)\frac{d(x, Sy)d(y, Sx)}{1 + d(x, Sy) + d(y, Sx) + d(x, y)};$$

then S have a unique fixed point.

Proof. Take $T = S$ in Theorem 2.2 □

Corollary 2.4. *Let (X, d) be a complex valued metric space and let $S, T : X \to X$. If there exists constants $\gamma > 0$ such that*

$$\gamma < 1;$$

and for all $x, y \in X$ we have

$$d(Sx, Ty) \precsim \gamma\frac{d(x, Sy)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(x, y)};$$

then S and T have a unique common fixed point.

Proof. Take γ a constant functions in Theorem 2.2. □

Example 2.5. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{C}$

$$d(x, y) = |x - y| + i|x - y|$$

for all $x, y \in X$. Then (X, d) is a complex metric space. Now we define the mappings $S, T : X \to X$ by

$$S(x) = \frac{x}{6} \text{ and } T(y) = \frac{y}{6}.$$

Consider the functions $\gamma : X \times X \to [0, 1)$

$$\gamma(x, y) = \frac{x^2y^2}{30}.$$

Clearly $\gamma(x_0, x_1) < 1$.
We satisfy the condition (a)of main theorem 2.2 as follows.

$$\begin{aligned}\gamma(TSx, y) =&\gamma(T(\frac{x}{6}), y) = \gamma(\frac{x}{36}, y)\\ &\leq \gamma(x, y),\end{aligned}$$

That is $\gamma(TSx, y) \leq \gamma(x, y)$, for all $x, y \in X$.
And

$$\begin{aligned}\gamma(x, STy) =&\gamma(x, S(\frac{y}{6})) = \gamma(x, \frac{y}{36})\\ &\leq \gamma(x, y),\end{aligned}$$

That is $\gamma(x, STy) \leq \gamma(x, y)$, for all $x, y \in X$.

Now for the verification of condition (c), we have for all $x, y \in X$

$$0 \precsim \frac{d(x, Sx)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(y, x)}.$$

Consider

$$\begin{aligned}d(Sx, Ty) =&d(\frac{x}{6}, \frac{y}{6}) = |\frac{x}{6} - \frac{y}{6}| + i|\frac{x}{6} - \frac{y}{6}|\\ =&\frac{1}{6}(|x - y| + i|x - y|)\\ \precsim \gamma(x, y)&\frac{d(x, Sx)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(y, x)}\end{aligned}$$

Therefore all the conditions of Theorem 2.2 are satisfied and $x = 0 \in X$ is a unique common fixed point of S and T.

3. Conclusion

This paper has explored the dynamic realm of fixed point theory, particularly within the intricate domain of contraction mappings in complex metric spaces. By introducing the concept of control functions, we have shed new light on the behavior and properties of fixed points, enriching our understanding of their convergence properties. Our investigation highlights the symbiotic relationship between contraction mappings and complex metric spaces, underscoring the indispensable role of control functions in shaping the trajectory of fixed-point iterations. Through our analysis, we have not only advanced the theoretical framework of fixed point theory but also opened avenues for further exploration and application in various mathematical and scientific domains.

Acknowledgment

The author would like to thank to express their gratitude to the anonymous reviewers for their constructive feedback and the editor for their thorough evaluation of this manuscript.

Declarations

The author declares that he has no conflicts of interest.

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On the kernel conditional density estimator with functional explanatory variable

IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

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DOI: https://doi.org/10.69717/ijams.v2.i1.117

abstract

This article focuses on the relationship between a scalar-explained random variable Y and a functional explanatory random variable X. In fact, through this work, our aim is to estimate the conditional probability density $f(y/x)$ when the explanatory variable X is functional using the kernel method. More precisely, we will present a numerical application based on simulated samples, with the aim of, on the one hand, highlighting the implementation of the estimator in question and the impact of using a symmetric kernel on its quality. On the other hand, when analyzing the performance of this estimator as a function of the sample size, the hypothesis imposed on the smoothing parameters (the smoothing parameters in the X direction and the Y direction are independent and the smoothing parameter in the X direction is the same as that in the Y direction) and the norm used in its construction.

keywords

Conditional density, Functional explanatory variables, Norms, Errors, Simulation.

2020 Mathematics Subject Classification

Primary 62G05; 62G07 · Secondary 62R10. ·

1. Introduction

The first work on the nonparametric kernel conditional density estimation, when the explanatory variable is functional, was introduced by Ferraty et al. A. Ferraty and Vieu, 2006; F. Ferraty, 2006. With the same approach followed by Rosenblatt Rosenblatt, 1969, for the real explanatory variable, the authors have constructed and analyzed the kernel conditional density estimator in the functional explanatory variable. Since these two works, the literature has developed on the kernel estimation of the conditional density in the framework of functional explanatory variables, its derivatives, and its applications in other fields, we can cite for example the works of: Ezzahrioui and Ould Saïd M. Ezzahrioui, 2010; M. Ezzahrioui and E, 2005, Ezzahrioui E. [N. Ezzahrioui, 2008, Laksaci A. Laksaci and Mechab, 2010, Laksaci and Mechab A. N. Laksaci, 2007, Ferraty et al. A. Ferraty and Vieu, 2008; F. Ferraty et al., 2010 and Dabo-Niang Dabo-Niang, 2007.

Let (X, Y) be a couple of random variables in $\mathbb{F}\times\mathbb{R}$, where $(\mathbb{F}, \|.\|)$ is a functions space equipped with a norm $\|.\|$, i.e X is a (time-dependent) functional random variable depending in time ($X \equiv X(t)$). Let $(X_i, Y_i)_{1\leq i\leq n}$ be n independent pairs, identically distributed as the couple (X, Y) and (x, y) be a fixed element of $\mathbb{F}\times\mathbb{R}$. The kernel estimator of $f(y/x)$ in this context is defined by:

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Communicated Editor: A. Necir
Manuscript received *Jan 22, 2025;* ***revised*** *April 15, 2025;* ***accepted*** *May 25, 2025;* ***published on line*** *Jun 02, 2025..*

$$\hat{f}_{ab}(y/x) = \frac{\sum_{j=1}^{n} K_1\left(\frac{\|x-X_j\|_p}{a}\right) K_2\left(\frac{y-Y_j}{b}\right)}{b\sum_{j=1}^{n} K_1\left(\frac{\|x-X_j\|_p}{a}\right)}, \tag{1}$$

where K_1 is a real asymmetric kernel function on $\mathbb{R}_+$, K_2 is a real symmetric kernel function on $\mathbb{R}$, $a > 0$ and $b > 0$ are the smoothing parameters in the X and Y directions respectively, and $\|.\|_p$ is a p-distance defined on $\mathbb{F}$ (for more details on the p-distance see Section 2).

In order to establish the mean square convergence of the estimator (1), Laksaci A. N. Laksaci, 2007 defined the formulas for bias and variance and the asymptotic square error of the estimator. Also, Ferraty et al. F. Ferraty, 2006, showed the both pointwisely and uniformly almost complete convergence of the estimator in question. Ferraty et al. A. Ferraty and Vieu, 2006 estimated the J^{th} order of derivative of the estimator 1.

To simplify the formula (1), we propose another version, similar to that used by Youndjé Youndjé, 1996 in the case of one explanatory variable, under the hypothesis that the two smoothing parameters a and b are equal ($h = a = b$). So, the expression (1) can be rewritten as follows:

$$\hat{f}_{h}(y/x) = \frac{\sum_{j=1}^{n} K_1\left(\frac{\|x-X_j\|_p}{h}\right) K_2\left(\frac{y-Y_j}{h}\right)}{h\sum_{j=1}^{n} K_1\left(\frac{\|x-X_j\|_p}{h}\right)}, \tag{2}$$

2. Concept of the norm

To study data, we often need to have a notion of distance between them. In mathematics, a distance (or metric) $d(.,.)$ is an application that formalizes the intuitive idea of distance, i.e. it represents the length that separates two points.

One of the most popular idea in mathematics to calculate a distance between two points is to use a $p-$norm that we denote by $\|.\|_p$. Thus, to study the estimators introduced in the previous section, it is interesting to recall some common norms used in such estimators.

Let $\mathbb{F}$ be a set of functions and $x(t)$ and $y(t)$ two functions defined in $\mathbb{F}$. The most commonly used norms in practice to measure the distance between these two functional points (these two functions) are presented in the following table:

Name	Parameter	Expression
Manhattan norm	1-norm	$\int_{\mathbb{F}} \lvert x(t) - y(t)\rvert\, dt$
Euclidian norm	2-norm	$\left(\int_{\mathbb{F}} (x(t) - y(t))^2 dt\right)^{\frac{1}{2}}$
Minkowski norm	p-norm	$\left(\int_{\mathbb{F}} \lvert x(t) - y(t)\rvert^p\, dt\right)^{\frac{1}{p}}$
Tchebychev norm	∞-norm	$\lim_{p\to\infty} \left(\int_{\mathbb{F}} \lvert x(t) - y(t)\rvert^p\, dt\right)^{\frac{1}{p}} = \sup_t \lvert x(t) - y(t) \rvert$

Table 1: Some norms which are generally used as distance in the functional framework.

3. Choice of kernel and smoothing parameter

From the two expressions (1) and (2), it is clear that the implementation of these estimators relies on the prior fixing of the kernel, the smoothing parameter, and the norm $\|.\|_p$. In our work, we focus particularly on the problem of choosing the smoothing parameter, because the choice of the kernel function remains the same as the univariate density. Furthermore, because $\|u\|_p$ is always a positive quantity, the real kernel K_1 should have positive support, consequently, we must use asymmetric density functions for the kernel K_1 (see Chen, 1999, 2000). While K_2 we must use a symmetric density functions because $\frac{y-Y_j}{h} \in \mathbb{R}$ (see Silverman, 2018). However, the problem of choosing the smoothing parameter has received serious attention because the numerical and graphical characteristics of the designed kernel estimator are very sensitive to the variation of the smoothing parameter, where small values of this parameter (compared to the optimal smoothing parameter) generate the phenomenon of under-smoothing, while large values of this parameter generate the phenomenon of over-smoothing.

Similarly to the cross-validation approach followed by Youndjé Youndjé, 1996 when the explanatory variable is scalar, Rachdi and Vieu Rachdi and Vieu, 2007 and Benhenni et al. Benhenni et al., 2007 proposed, respectively, a global leave-out-one-curve and a local adaptive leave-out-one-curve cross-validation procedure for the regression operator estimation in functional data. Laksaci et al. A. Laksaci et al., 2013 constructed the global and local leave-out-one-curve cross-validation procedures in the context of conditional density when the explanatory variable is functional.

Global and local bandwidth selection rules

The idea of this approach is based on minimizing the integrated squared error, which is weighted by the probability measure, $dP_X(x)$, of the functional variable X and some non-negative weighting functions W_1 and W_2 associated to the variables x and y respectively. That is to say, they considered the integrated squared error defined by the following expression:

$$ISE(\hat{f}, f) = \int\int \left(\hat{f}(y/x) - f(y/x)\right)^2 W_1(x)W_2(y)dP_X(x)dy. \tag{3}$$

So, the mean integrated squared error will be given as follows:

$$MISE(\hat{f}, f) = \int\int \mathbb{E}\left(\hat{f}(y/x) - f(y/x)\right)^2 W_1(x)W_2(y)dP_X(x)dy. \tag{4}$$

Discretizing the expression (3) allows us to obtain an approximation of the mean square error given by:

$$ISE(\hat{f}, f) \approx \frac{1}{n}\sum_{i=1}^{n}\left(\hat{f}(Y_i/X_i) - f(Y_i/X_i)\right)^2 \frac{W_1(X_i)W_2(Y_i)}{f(X_i, Y_i)}. \tag{5}$$

Concerning the weighting functions W_1 and W_2, we recall that these functions were introduced to reduce bounds effects thanks to their support. But, in practice, Härdle and Marron Härdle and Marron, 1985 have emphasized that the role of their expressions is not very determining and that they are functions arbitrarily chosen by the user. Laksaci et al. A. Laksaci et al., 2013 took the expressions of W_1 and W_2 in their simulation as:

$$W_2(z) = \begin{cases} 1 & if\ z \in [\min Y_i, i = 1...n \times 0.9 \max Y_i, i = 1...n \times 1.1] \\ 0 & otherwise, \end{cases}$$

and

$$W_1(t) = \begin{cases} 1 & if\ \min d(t, X_i) < a_0 \\ 0 & otherwise, \end{cases}$$

where $a_0 = \min(a_q)$ and a_q is the quantile of order q of the vector of all distances between the curves.

We note that the ISE and $MISE$ functions depend on the unknown conditional density f, so, in practice, the smoothing parameters that minimize these errors are not computable. By following the same ideas as in Youndjé Youndjé, 1996 for the real case, Laksaci et al. A. Laksaci et al., 2013 proposed another function that is asymptotically equivalent to the quadratic distance given in (3). Where they suggest to replace (3) by the integrated squared error expressed as follows:

$$ISE(\hat{f}, f) = A + B - 2C,$$

where

$$A = \int\int \hat{f}(y/x)^2 W_1(x)W_2(y)dP_X(x)dy,$$
$$B = \int\int f(y/x)^2 W_1(x)W_2(y)dP_X(x)dy,$$

and

$$C = \int\int \hat{f}(y/x)f(y/x)W_1(x)W_2(y)dP_X(x)dy.$$

Since the second term B is independent of the smoothing parameter (a, b), the problem of minimizing the ISE is equivalent to that of minimizing the function $A - 2C$. Thus, to select the bandwidth (a, b) that minimizes the approximate ISE ($AISE = A - 2C$), we must first estimate the two quantities A and C whose form are as follows;

$$\begin{aligned} C &= \int\int \hat{f}(y/x)f(y/x)W_1(x)W_2(y)dP_X(x)dy, \\ &= \int\int \hat{f}(y/x)W_1(x)W_2(y)dP_{Y/X=x}(y)dP_X(x), \\ &= \int\int \hat{f}(y/x)W_1(x)W_2(y)dP_{(X,Y)}(x,y), \\ &= \mathbb{E}_{(X,Y)}\left(\hat{f}(Y/X)W_1(X)W_2(Y)\right), \end{aligned}$$

and,

$$A = \mathbb{E}_X\left(\int \hat{f}(y/X)W_1(X)W_2(y)dy\right),$$

where $\mathbb{E}_X$ denotes the mean associated with the distribution of the random variable X.

For the aim to minimize the function $A - 2C$, the authors have followed the idea of Rudemo Rudemo, 1982 and Rachdi and Vieu Rachdi and Vieu, 2007 where they adopted the cross-validation technique with leave-out-one-curve principle. More precisely, they constructed the following criteria, for the global smoothing parameter:

$$\begin{aligned} GCV(a,b) &= \frac{1}{n}\sum_{i=1}^{n} W_1(X_i)\int\left(\hat{f}_{-i}\left(y/X_i\right)\right)^2 W_2(y)dy \\ &\quad -\frac{2}{n}\sum_{i=1}^{n}\hat{f}_{-i}\left(Y_i/X_i\right)W_1(X_i)W_2(Y_i), \end{aligned} \tag{6}$$

and the following local criteria, for a fixed $y \in \mathbb{R}$ and $x \in \mathbb{F}$;

$$\begin{aligned} LCV(a,b) &= \frac{1}{n}\sum_{i=1}^{n} W_{1,x}(X_i)\int\left(\hat{f}^2_{-i}\left(z/X_i\right)\right) W_{2,y}(z)dz \\ &\quad -\frac{2}{n}\sum_{i=1}^{n}\hat{f}_{-i}\left(Y_i/X_i\right)W_{1,x}(X_i)W_{2,y}(Y_i), \end{aligned} \tag{7}$$

where, $W_{2,x}$ (respectively $W_{2,y}$) is some positive local weight function around x (respectively y), Laksaci et al. A. Laksaci et al., 2013 used the following local weight functions:

$$W_{1,x} = \begin{cases} 1 \; if \; \min d(t;x) < a(x); \\ 0 \; otherwise \end{cases} \quad \text{and} \quad W_{2,y}(z) = \begin{cases} 1 \; if \; |z-y| < b(y); \\ 0 \; otherwise \end{cases}$$

where $a(x)$ (respectively, for $b(x)$) the ball centered at x (respectively the interval centered at y) with radius $a(x)$ (respectively with radius $b(y)$) contains exactly k neighbors of x (respectively of y).

And $\hat{f}_{-i}(y/x)$ represent the kernel conditional density estimator, using the cross-validation technique computed from the set of points except the point (x_i, y_i), its formula is given by:

$$\hat{f}_{-i}(y_i/x_i) = \frac{\sum\limits_{j=1,j\neq i}^{n} K_1\left(\frac{\|x_i - X_j\|_p}{a}\right)K_2\left(\frac{y_i - Y_j}{b}\right)}{b\sum\limits_{j=1,j\neq i}^{n} K_1\left(\frac{\|x_i - X_j\|_p}{a}\right)},$$

Hence, the global (respectively, local) cross-validation procedure consists of choosing the smoothing parameters (a, b) which minimize the criteria GCV (respectively LCV).

4. Numerical application

This section aims to illustrate, via numerical examples and using the simulation approach, how to implement the conditional kernel density estimator when the explanatory variable is functional, and to verify the impact of the substitution of the kernel K_1, initially asymmetric, and the kernel K_2 by a same symmetric kernel K. Also, we focus in the choice of the norm used to calculate the distance between the points $x(t)$ and $x_i(t)$ on the quality of the designed estimator.

4.1. Presentation of the application and its parameters

In order to study the effect of using a symmetric kernel in the performance of kernel conditional density estimation when the explanatory variable is functional, let consider that $K_1 = K_2 = K$, with K is a real symmetric kernel function on $\mathbb{R}$. that is, we rewrite (1) and (2) respectively as follows:

$$\hat{f}_{ab}(y/x) = \frac{\sum_{j=1}^{n} K\left(\frac{\|x-X_j\|_p}{a}\right) K\left(\frac{y-Y_j}{b}\right)}{b\sum_{j=1}^{n} K\left(\frac{\|x-X_j\|_p}{a}\right)}, \tag{8}$$

and

$$\hat{f}_{h}(y/x) = \frac{\sum_{j=1}^{n} K\left(\frac{\|x-X_j\|_p}{h}\right) K\left(\frac{y-Y_j}{h}\right)}{h\sum_{j=1}^{n} K\left(\frac{\|x-X_j\|_p}{h}\right)}, \tag{9}$$

In order to meet our objective, we have implemented a simulator in a *Matlab* environment, whose main steps are:

1. Generate m samples $(X_i^{(l)}, Y_i^{(l)})$ of size n of a target distribution, where $l = 1, ..., m$ and $i = 1, ..., n$.
2. Compute $(\hat{a}, \hat{b})$ and $\hat{h}$ that minimize the average of the ISE associated with each estimator.
3. Calculate the both estimators (8) and (9) and compare their performance.

To realize these steps, and for calculation reasons, we proposed to discretize (to approximate) the average ISE. More precisely, we use the discretized expression of the average ISE that is given as follows: (see Bashtannyk and Hyndman, 2001):

$$\overline{ISE} = \frac{\Delta}{nN}\sum_{l=1}^{m}\sum_{j=1}^{J}\sum_{i=1}^{n}\left[\hat{f}(y_j'/x_i^{(l)}) - f(y_j'/x_i^{(l)})\right]^2, \tag{10}$$

where $(x_i, y_i), i = 1, ..., n$ an independent and identically distributed observations from the joint density of (X, Y), $y' = \left(y_1', y_2', ..., y_J'\right)$ is a vector of equidistant points in the space of Y and $\Delta = y_{j+1}' - y_j'$, $\forall j \in \{1, 2, ..., J-1\}$.

Consequently, the estimators of the optimal smoothing parameters, in the sense of the average of the ISE, correspond to the quantities which minimize the expression (10).

For the simulation example, we considered the following model, which represents a conditional density Y knowing $X = x$ follows a normal law with mean $\|x\|_2$ and variance 1, given by:

$$f(y/x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y-\|x\|_2\right)^2}, \tag{11}$$

and suppose that the explanatory variables X_i from a stochastic process similar to that proposed by Delsol Delsol, 2008, and it defined by:

$$X_i = X_i(t) = a_i\cos(2\pi t) + b_i\sin(3\pi t) + c_i(t-0.45)(t-0.75)e^{(-d_i t)},$$

with $t \in [0, 1]$, $a_i \rightsquigarrow N(-1, 1)$, $b_i \rightsquigarrow N(-1, 1)$, $c_i \rightsquigarrow U[1, 5]$ and $d_i \rightsquigarrow U[1, 5]$, where N and U respectively designate a normal distribution and a uniform distribution.

An illustration example about the variation of our model which defined in (11), depending on $\|x\|_2$ is presented in figure 1. The Figure 2 represents an example of a sample of size $n = 10$ from the variable X, where the red curve in the figure 2 represents the theoretical $E(X(t))$, that is to say when $E(a) = E(b) = -1$ and $E(c) = E(d) = 3$.

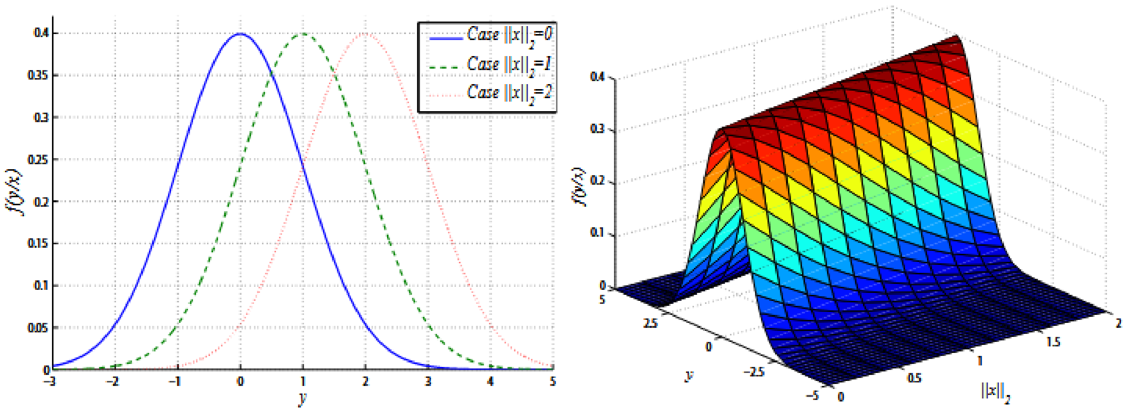

Figure 1: Illustration curves about the variation of the density $f(y/x)$ in depending on $\|x\|_2$.

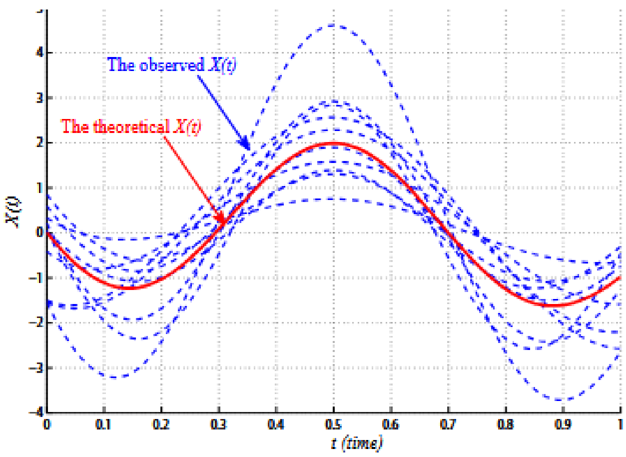

Figure 2: Ten examples of a sample xi(t) generated from the variable X.

4.2. Numerical and graphic results

To obtained the results of the numerical application, we consider to use the Gaussian kernel and the norm $p \in \{1,\ 2,\ \infty\}$ in the construction of the two versions of the conditional density estimator in question. The application was carried out on 100 samples ($m = 100$) of size $n \in \{50\ ,\ 100\ ,\ 200\ ,\ 500\ ,\ 1000\ ,\ 2000\}$ at the point $x = E(X)$ (see figure 2).

The numerical results obtained in our application are arranged in Table 2 and are presented in Figures 3–4.

Norm	n	$a = b$		$a \neq b$	
		$\hat{h}$	$\overline{ISE}_h$	$(\hat{a}, \hat{b})$	$\overline{ISE}_{(a;b)}$
$\lVert.\rVert_1$	50	0.8739	0.0187	(23.7655 ; 0.5509)	0.0064
	100	0.8707	0.0176	(32.6362 ; 0.4953)	0.0043
	200	0.8689	0.0166	(23.3421 ; 0.419)	0.0023
	500	0.8681	0.0162	(31.3126 ; 0.3695)	0.0014
	1000	0.8672	0.0162	(28.8502 ; 0.3176)	0.001
	2000	0.8665	0.0161	(35.6687 ; 0.2659)	0.0006
$\lVert.\rVert_2$	50	0.8771	0.0176	(23.6759 ; 0.5784)	0.0062
	100	0.8703	0.0173	(21.494 ; 0.5151)	0.0039
	200	0.867	0.0173	(17.8311 ; 0.4541)	0.003
	500	0.8668	0.0165	(30.8016 ; 0.348)	0.0014
	1000	0.8667	0.016	(33.0062 ; 0.3195)	0.0008
	2000	0.8662	0.0156	(39.1335 ; 0.2431)	0.0005
$\lVert.\rVert_\infty$	50	0.8817	0.0209	(28.8429 ; 0.5458)	0.0067
	100	0.8747	0.0177	(23.7255 ; 0.4911)	0.0041
	200	0.8674	0.0176	(26.9714 ; 0.4362)	0.0024
	500	0.8676	0.0165	(29.9083 ; 0.3765)	0.0016
	1000	0.8663	0.0164	(37.9307 ; 0.2872)	0.0009
	2000	0.8655	0.0164	(41.8221 ; 0.2384)	0.0006

Table 2: Variation of $\overline{ISE}$ according to the sample size n, the norm $\lVert.\rVert_p$ and the hypothesis imposed on the smoothing parameters.

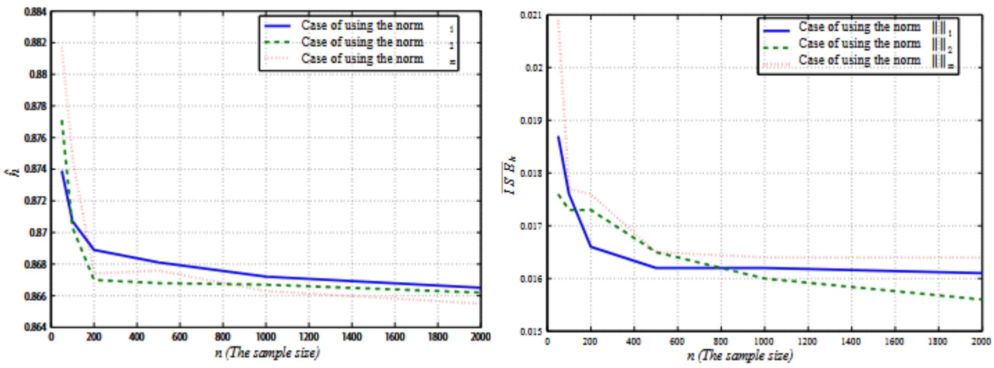

Figure 3: Variation of $\hat{h}$ and $\overline{ISE}_{\hat{h}}$ according to the sample size.

4.3. Discussion of results

Taking into account the numerical and graphical results obtained in the previous section, we note that:

- In all situations considered, the optimal smoothing parameters decrease where the sample size increases, which coincides with the following fundamental property (condition) of the smoothing parameter: $\lim_{n\to\infty} h(n) = 0$.
- Independently of the norm used, the estimators $\hat{f}_{ab}(y/E(X))$ and $\hat{f}_h(y/E(X))$ converge to $f(y/E(X))$ in L_2 (average ISE) and this can be justified by the decrease of the average ISE (convergence to zero), associated with the estimators in question, as the sample size n increases..
- The estimator $\hat{f}_{ab}(y/E(X))$ is more efficient, in the sense of the average ISE, than the estimator $\hat{f}_h(y/E(X))$ and this independently of the sample size and the norm used for the construction of these two estimators.
- The three norms used practically provide us with estimators of the same performance (average ISE). But in general, we see that there is a slight preference:
  - For the $\|.\|_2$ norm when the sample size is very small.
  - For the $\|.\|_1$ norm when the sample size is medium.
  - For the $\|.\|_2$ norm when the sample size is large.
- Because of the positivity of the distance between x and x_i ($\|x - x_i\|_p \geq 0$), the kernel must be defined on a support positive when the explanatory variable is functional. Our results show that even for a symmetric kernel we can have reasonable results.

5. Conclusion

By a small research in the literature, we can note that the conditional density when the explanatory variable is functional is a rich problem in statistics and it is in high demand in many fields of application, see Bosq, 2000, Bosq, 2000, F. Ferraty, 2006, Ramsay and Silverman, 2005. In this work, our objective was to present an illustrative numerical example of the implementation of the conditional density estimator, when the explanatory variable is functional, and we mainly focus on its performance as a function of the sample size, the hypothesis imposed on the smoothing parameters, and the norm used in its construction, and to highlight the impact of using a symmetric kernel on its quality.

Acknowledgment

The author would like to thank to express their gratitude to the anonymous reviewers for their constructive feedback and the editor for their thorough evaluation of this manuscript.

Declarations

The author declares that he has no conflicts of interest.

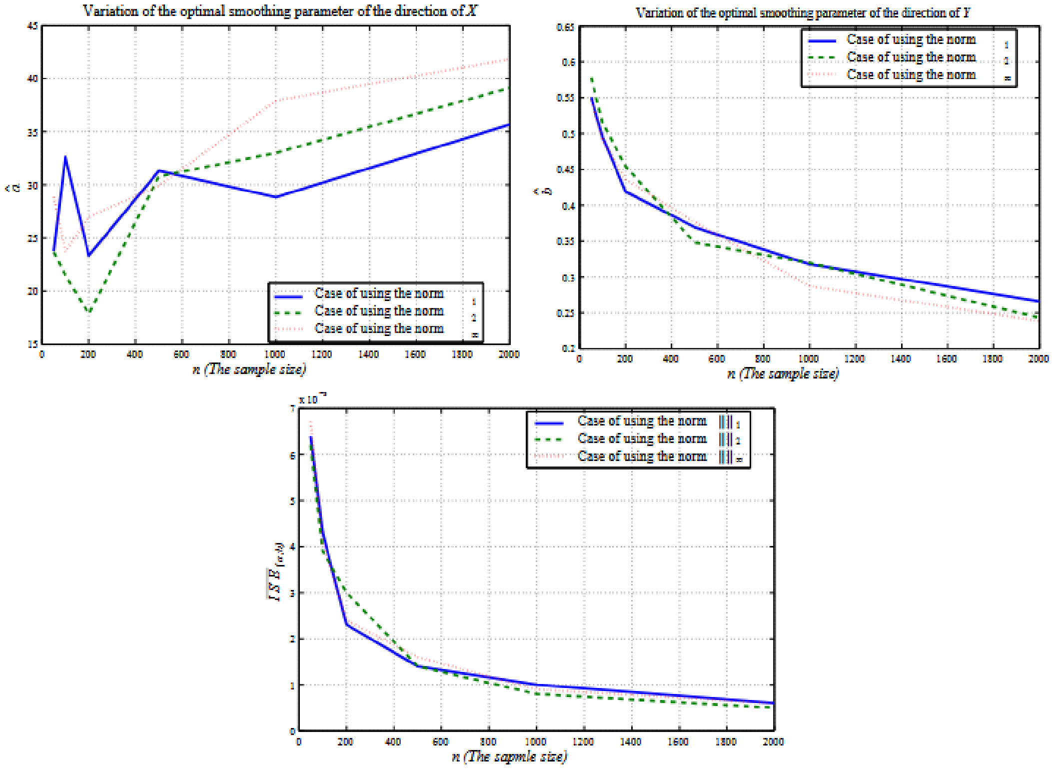

Figure 4: Variation of $(\hat{a}, \hat{b})$ and $\overline{ISE}_{(\hat{a},\hat{b})}$ according to the sample size.

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An effective operational matrix method for the solution of non-linear third-order initial value problems

IJAMS Peer-reviewed journal
International Journal of Applied Mathematics and Simulation

An effective operational matrix method for the solution of non-linear third-order initial value problems

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DOI: https://doi.org/10.69717/ijams.v2.i1.118

abstract

The present paper provides a new technique using the clique polynomials as basis function for the operational matrices to obtain numerical solutions of third-order non-linear ordinary differential equations. It aims to find all solutions as easy as possible. Numerical results derived using the proposed techniques are compared with the exact solution or the solutions obtained by other existing methods. The new numerical examples were examined to show that the new approach is highly efficient and accurate. The approximate solutions can be very easily calculated using computer program Matlab.

keywords

The clique polynomials; Operational matrix; Collocation points; Third-order of differential equations; Initial value problems.

2020 Mathematics Subject Classification

Primary 62G05; 62G07 · Secondary 62R10. ·

1. Introduction

Many problems in physics, chemistry, and engineering science are modeled as third-order boundary value problems or initial value problems. These boundary value problems can be found in different areas of applied mathematics and physics such as, in the deflection of a curved beam having a constant, thin-film flow, and gravity-driven flows (see Momoniat and Mahomed, 2010; Tuck and Schwartz, 1990). Most nonlinear differential equations do not have exact solutions, so approximation and numerical techniques must be used. Many researchers developed some methods to solve boundary and initial value problems of different order such as Agarwal, 1986; Butcher, 2016; Fatima, 2024 and others. In this paper, we focus on initial value problems of third-order nonlinear ordinary differential equations.

$$\begin{cases} y''' = f\left(x, y\left(x\right), y'\left(x\right), y''\left(x\right)\right) \\ y\left(x_0\right) = \alpha, y'\left(x_0\right) = \beta, y''\left(x_0\right) = \gamma, x \in \left[x_0, x_{end}\right] \end{cases} \tag{1}$$

where $y(x) \in R, f := R \times R \times R \times R \to R$ is a continuous function and α, β and γ are constants. Several direct methods are widely proposed by researchers in solving third-order differential equations such as iterative method, Traub's method Chun and Kim, 2010, block method Abu Arqub et al., 2013; Mehrkanoon, 2011; Yap et al., 2014, Runge-Kutta method Fang et al., 2014; Lee et al., 2020; You and Chen, 2013, operational matrices of Bernstein polynomials method Khataybeh et al., 2019; Malik et al., 2021 and more.
The main of this paper is to apply the new operational matrix of integration method using clique polynomials to solve the third-order initial value problems. It is shown that the method provides the solution in a rapid

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Communicated Editor: M. Berbiche
Manuscript received *Oct 27, 2024;* ***revised*** *April 24, 2025;* ***accepted*** *May 11, 2025;* ***published on line*** *June 14, 2025.*

convergent series. The other operational matrix method using clique polynomials has been used by Kumbinarasaiah et al., 2021 and Ganji et al., 2021 to solve effectively the non linear Klein Gordon equation and non-linear fractional Klein Gordon equation, which converge rapidly to accurate solutions. We show that the initial value problems of third-order can be solved efficiently using the clique polynomials. The present method converts Eq. (1) to a system of algebraic equations which can be solved easily. The capability of the method shall be tested on a linear and nonlinear third-order differential equations.
This paper is arranged as follows. In Section 2, we give the interesting properties of clique polynomials and there convergence analysis. In Section 3, we construct the operational matrix technique using the clique polynomials for solving numerically the nonlinear third-order differential equations. Section 4 includes to present several results and discussions to show the efficiency and simplicity of the proposed method. Finally, conclusion is given in Section 5.

2. Clique polynomials and convergence analysis

Let G be a graph that is free from multi edges and loops. The clique polynomial of a graph G, denoted by $C(G;x)$, is characterized by Hoede and Li, 1994

$$C(G;x)=\sum_{k=0}^{n}a_kx^k$$

where a_k represent the total distinct $k-$cliques in graph of size k, with $a_0=1$. The clique polynomial of a complete graph K_n with $n-$vertices is given by

$$C\left(K_n;x\right)=(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$$

where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$
In particular

$$\begin{aligned}C(K_0;x)&=1\\C(K_1;x)&=1+x\\C(K_2;x)&=1+2x+x^2\\C(K_3;x)&=1+3x+3x^2+x^3\end{aligned}$$

Let $B=\{C_n(x)=C(K_n,x),n\in N\}$. Clearly B is Banach space on closed subset A of R with norm given by

$$\|C_n\|=\sup_{\forall x\in A}|C_n\left(x\right)|\qquad\forall C_n\in B\left(A\right)$$

We can approximate any function $f(x)$ in $L^2[0,1]$ in terms of the clique polynomial as (see Ganji et al., 2021; Kumbinarasaiah et al., 2021)

$$f\left(x\right)\approx\widetilde{f}\left(x\right)=\sum_{i=0}^{n-1}a_iC\left(K_i;x\right)$$

We can write

$$f\left(x\right)=\sum_{i=0}^{n-1}a_i(\sum_{k=0}^{i}\binom{i}{k}x^k)=A^TPX\left(x\right)$$

where $A^T=[a_0,a_1,\ldots,a_{n-1}]$, $X(x)=[1,x,\ldots,x^{n-1}]^T$ and P is the lower triangular $n\times n$ matrices defined by

$$p_{ij}=\begin{cases}0 & j>i,i,j=1,2,...,n\\ \frac{(i-1)!}{(i-j)!(j-1)} & i\geq j,i,j=1,2,...,n\end{cases}$$

3. Description of the clique polynomial operational matrix method

We consider the clique polynomial operational matrix method along with collocation points to solve the following third-order of differential equations

$$y^{(3)}=f(x,y,y',y''),0\leq x\leq 1\tag{2}$$

with the initial conditions

$$y(0) = b_1, y'(0) = b_2, y''(0) = b_3 \tag{3}$$

where b_1, b_2, b_3 are real constants and f is a given continuous on $[0,1]$, nonlinear function. We assume that

$$y'''(x) = A^T P X(x) \tag{4}$$

Where A is an unknown vector to be determined $A^T = [a_0, a_1, \dots, a_{n-1}]$, $X(x)$ is the known vector defined above and

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \cdots & 0 \\ 1 & 3 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 1 & n-1 & \frac{(n-1)(n-2)}{2!} & \cdots & n-1 & 1 \end{bmatrix}$$

For solving the equation (2), we calcul the derivatives $y^{(k)}(x)$ where $k = 0, 1, 2, 3, x \in [0,1]$ and with the initial conditions (3)

It is easy to prove that this identity

$$\underbrace{\int_0^x \int_0^x \dots \int_0^x}_{k \text{ times}} A^T P X(t)\, dt = A^T P M_k x^k X(x)$$

where M_k is the $n \times n$ matrices

$$M_k = \begin{bmatrix} \frac{1}{k!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2\times 3\times \ldots (k+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3\times 4\times \ldots (k+2)} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n(n+1)\ldots(n+k-1)} \end{bmatrix}$$

Integrating equation (4) third times on bothside with respect to x limit between 0 and x, we obtain

$$y(x) = b_1 + b_2 x + \frac{b_3}{2} x^2 + \int_0^x \int_0^x \int_0^x A^T P X(t) dt$$

After integration yields

$$y(x) = b_1 + b_2 x + \frac{b_3}{2} x^2 + A^T P M_3 x^3 X(x)$$

where

$$M_3 = \begin{bmatrix} \frac{1}{3!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{4!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3\times 4\times 5} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n(n+1)(n+2)} \end{bmatrix}$$

Now by substituting y, y', y'', y''' into equation (2) and collocate this equation by the following collocation points $x_i = \frac{2i-1}{2n}, i = 1, ..., n$, we get a system of n non linear equations with n unknowns $(a_0, a_1, ..., a_{n-1})$. The unknown coefficients are determined by satisfying the remaining the initial conditions (3) at chosen collocation points. This system can be solved by using the Newton method.

4. Numerical results

In order to test the proposed method, we present some numerical results obtained by applying operational matrix method to find numerical approximations of the solutions of some test problems $\left(x_i = \frac{1}{10} i; i = 0, 1, ..., 10\right)$. We will discuss the new numerical examples of third-order initial value problems. The tables 1-4 clearly show the improvements we achieved if compared to the exact solution. Figures 1, 3 and 5 show the comparison between the numerical solutions and the exact solutions of the initial value problems (Examples 1-3). Examining these tables, it is clear that the absolute errors were seem to be small. It is should be noted that the

Table 1: Numerical results for Example 1 ($n = 10$)

x	Exact solution	Numerical solution	Errors
0	-1	-1	0
0.1	-0.994995834721974	-0.994995834723177	$1.20281562487889E-12$
0.2	-0.979933422158758	-0.979933422162680	$3.92197385679083E-12$
0.3	-0.954663510874394	-0.954663510881710	$7.31581462076747E-12$
0.4	-0.918939005997115	-0.918939006007955	$1.08402176124400E-11$
0.5	-0.872417438109627	-0.872417438122700	$1.30726540703563E-11$
0.6	-0.814664385090322	-0.814664385113334	$2.30125918321278E-11$
0.7	-0.745157812715512	-0.745157812779116	$6.36040109469604E-11$
0.8	-0.663293290652835	-0.663293290813850	$1.61015867305991E-10$
0.9	-0.568390031729336	-0.568390032047044	$3.17708304109487E-10$
1	-0.459697694131860	-0.459697694888935	$7.57074403168190E-10$

approximate solution approaches the exact solution as n, the number of the basis functions, increases. All numerical computations have been done in Matlab (see Matlab program below), the program execution time by this method is 47 second.
Where

$$\text{Absolute error} = |\text{Exact solution} - \text{Numerical solution}|$$

Example 1 Consider the linear third-order initial value problem

$$y''' = sin(x), 0 \leq x \leq 1 \tag{5}$$

with initial conditions

$$y(0) = -1, y'(0) = 0, y''(0) = 1 \tag{6}$$

The analytic solution of the above problem is

$$y = cos(x) + x^2 - 2 \tag{7}$$

We have

$$y(x) = -1 + \frac{1}{2}x^2 + A^T P M_3 x^3 X(x)$$

Substituting equation (4) into (5) yields

$$A^T P X(x) = sin(x)$$

We collocate this equation at the collocation points $x_i = \frac{2i-1}{2n}, i = 1, ..., n$ to obtain numerical values of y. By using the conditions (6), the obtained system is solved, yielding the following results for $n = 10$

$$A = \begin{bmatrix} -0.810695 \\ 0.332789 \\ 1.038764 \\ -1.157273 \\ 1.142639 \\ -0.856889 \\ 0.418821 \\ -0.129569 \\ 0.023261 \\ -0.001849 \end{bmatrix}$$

Table 1 and 2 show that the numerical solutions and the errors obtained for linear third-order initial value problem (5) (Example 1) by using the present method and compared with the exact solution (7) for $n = 10$ and $n = 15$ respectively. Figure 1 shows the comparison between the approximate solution and the exact solution (7) of the problem (5). In Figure 2, the absolute errors have been shown at distinct points.

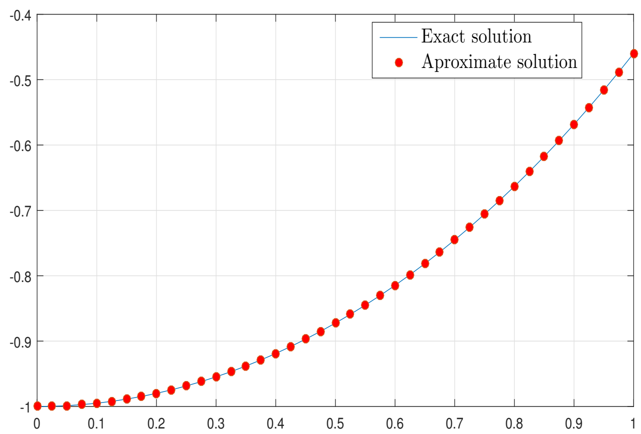

Figure 1: Comparison of approximate and exact solution for Example 1.

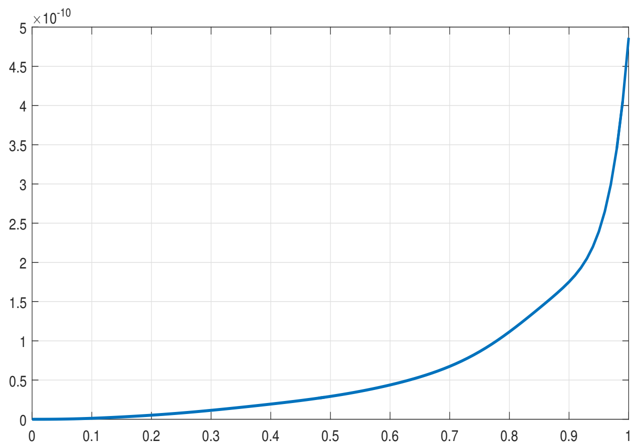

Figure 2: Error Analysis of Example 1.

Table 2: Numerical results for Example 1 ($n = 15$)

x	Exact solution	Numerical solution	Errors
0	-1	-1	0
0.1	-0.994995834721974	-0.994995834721283	$-9.7666319476275E-13$
0.2	-0.979933422158758	-0.979933422154582	$-6.9754202414174E-12$
0.3	-0.954663510874394	-0.954663510866042	$-2.57520671453904E-11$
0.4	-0.918939005997115	-0.918939005974870	$-6.47839559775321E-11$
0.5	-0.872417438109627	-0.872417438040750	$-1.19521503805231E-10$
0.6	-0.814664385090322	-0.814664384930306	$-1.78378645188104E-10$
0.7	-0.745157812715512	-0.745157812411764	$-2.44772868640553E-10$
0.8	-0.663293290652835	-0.663293290068459	$-3.49193451931740E-10$
0.9	-0.568390031729336	-0.568390030578339	$-5.29264299053978E-10$
1	-0.459697694131860	-0.459697691704301	$-7.88076048863218E-10$

Example 2 Consider the linear third-order initial value problem

$$y''' = 8e^{2x} + 2, 0 \leq x \leq 1 \tag{8}$$

with initial conditions

$$y(0) = -2, y'(0) = 2, y''(0) = 4 \tag{9}$$

The analytic solution of the above problem is

$$y(x) = e^{2x} + \frac{1}{3}x^3 - 3 \tag{10}$$

By solving the equation (8) with conditions (9) we obtain the vector A for $n = 10$

$$A = \begin{bmatrix} 3.099290 \\ 2.071009 \\ 2.360541 \\ 1.332371 \\ 0.476479 \\ 0.862816 \\ -0.466501 \\ 0.338103 \\ -0.090077 \\ 0.015966 \end{bmatrix}$$

Table 3 shows that the approximate solutions and the errors obtained for linear third-order initial value problem (8) (Example 2) and compared with the exact solution (10) for $n = 10$. Figure 3 shows the comparison between the approximate solution and the exact solution of the problem (8). Figure 4 shows the error Analysis of Example 2.

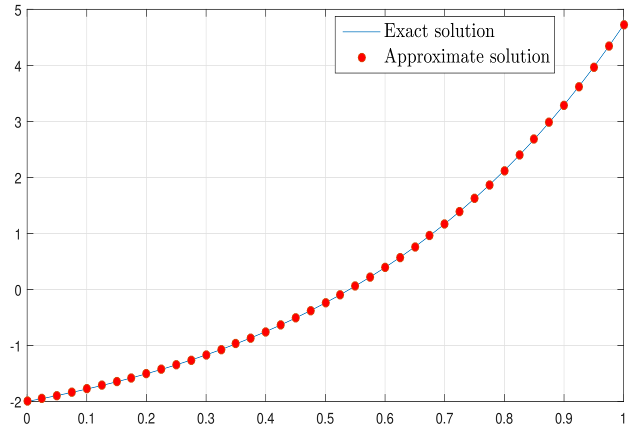

Figure 3: Comparison of approximate and exact solution for Example 2.

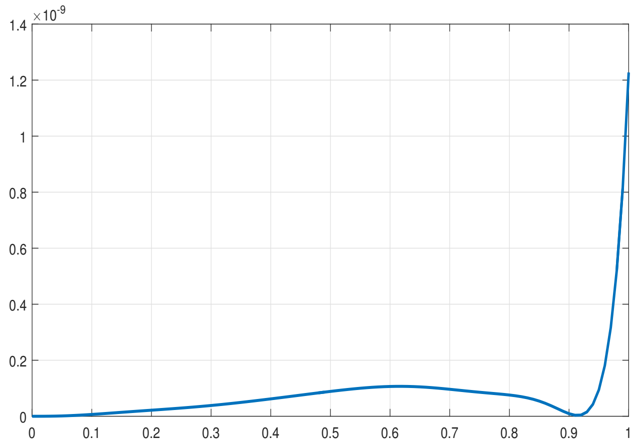

Figure 4: Error Analysis of Example 2.

Table 3: Numerical results for Example 2 ($n = 10$)

x	Exact solution	Numerical solution	Errors
0	-2	-2	0
0.1	-1.778263908506500	-1.77826390851309	$6.58939569575523E-12$
0.2	-1.505508635692060	-1.50550863571379	$2.17246221012601E-11$
0.3	-1.168881199609490	-1.16888119964800	$3.85114162781970E-11$
0.4	-0.753125738174199	-0.753125738236310	$6.21112050680495E-11$
0.5	-0.240051504874288	-0.240051504963057	$8.87692142015339E-11$
0.6	$+0.392116922736547$	$+0.392116922630138$	$1.06409547839803E-10$
0.7	$+1.169533300178010$	$+1.169533300081640$	$9.63700230727227E-11$
0.8	$+2.123699091061780$	$+2.123699090985880$	$7.59063922828318E-11$
0.9	$+3.292647464412950$	$+3.292647464403680$	$9.26281273905261E-12$
1	$+4.722389432263980$	$+4.722389431036580$	$1.22740306807145E-09$

Table 4: Numerical results for Example 3

x	Exact solution	Numerical solution for $n = 7$	Numerical solution for $n = 10$
0	1	1	1
0.1	0.90483741803590	0.90483741804721	0.90483741803568
0.2	0.81873075307798	0.81873075312857	0.81873075307623
0.3	0.74081822068171	0.74081822079209	0.74081822067633
0.4	0.67032004603563	0.67032004622430	0.67032004602522
0.5	0.60653065971263	0.60653065999590	0.60653065969547
0.6	0.54881163609402	0.54881163648641	0.54881163606573
0.7	0.49658530379141	0.49658530430750	0.49658530374543
0.8	0.44932896411722	0.44932896476990	0.44932896404636
0.9	0.40656965974059	0.40656966054202	0.40656965963669
1	0.36787944117144	0.36787944214112	0.36787944102411

Example 3 Consider the non-linear third-order initial value problem

$$y''' + y'' + y'y = -e^{-2x}, 0 \leq x \leq 1 \tag{11}$$

with initial conditions

$$y(0) = 1, y'(0) = -1, y''(0) = 1 \tag{12}$$

The analytic solution of the above problem is

$$y(x) = e^{-x} \tag{13}$$

By solving the equation (11) with conditions (12) we obtain the vector A for $n = 10$

$$A = \begin{bmatrix} -2.197399 \\ -0.840032 \\ 9.378083 \\ -18.330639 \\ 20.881442 \\ -15.525444 \\ 7.625636 \\ -2.391506 \\ 0.434777 \\ -0.034917 \end{bmatrix}$$

Table 4 and 5 show that the numerical solutions and the errors obtained for non-linear third-order initial value problem (11) (Example 3) and compared with the exact solution (13) for $n = 7$ and $n = 10$ respectively. Figure 5 shows the comparison between the approximate solution and the exact solution of the problem (11). Figure 6 shows the error Analysis of Example 3.

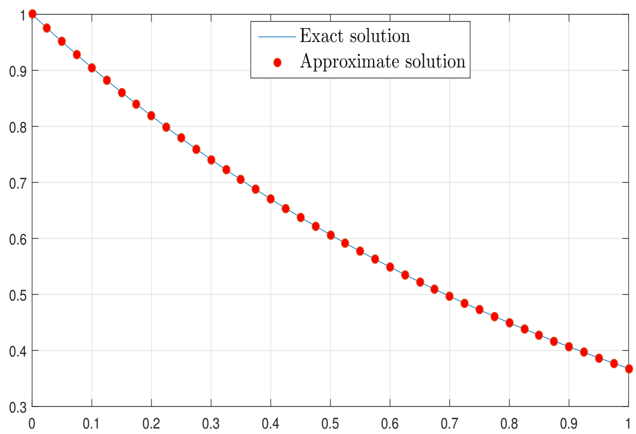

Figure 5: Comparison of approximate and exact solution for Example 3.

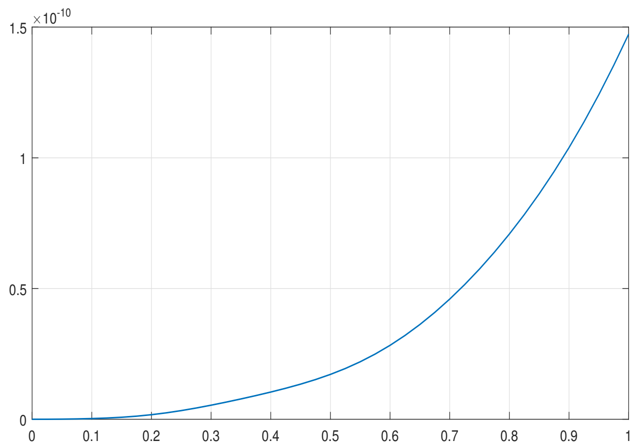

Figure 6: Error Analysis of Example 3.

Table 5: Errors obtained for Example 3

x	Errors for $n=7$	Errors for $n=10$
0	0	0
0.1	$1.12592157819336E-11$	$2.74891220897189E-13$
0.2	$5.05927522098659E-11$	$1.74982250911171E-12$
0.3	$1.10377484929813E-10$	$5.38702416008618E-12$
0.4	$1.88661086752973E-10$	$1.04127817479593E-11$
0.5	$2.83306489379243E-10$	$1.71629377376803E-11$
0.6	$3.92390342440763E-10$	$2.82878165336342E-11$
0.7	$5.16110376658219E-10$	$4.59714488698637E-11$
0.8	$6.52678022738939E-10$	$7.08584302344661E-11$
0.9	$8.01421373708422E-10$	$1.03907771276113E-10$
1	$9.69684443852259E-10$	$1.47327927635388E-10$

Table 6: Numerical results for Example 4 ($n=10$ and $B=1$)

x	Hb. method Adesanya et al., 2013	Bp. method Khataybeh et al., 2019	(CP) method
0.1	0.004999979166110	0.0049999583341723	0.004999958453095
0.2	0.019998666668590	0.0199986668419935	0.019998667405196
0.3	0.044998481293978	0.0449898794745896	0.044989880928476
0.4	0.079991467388617	0.0799573779857994	0.079957380252171
0.5	0.124967454367055	0.1248700575229549	0.124870060064380
0.6	0.179902837409194	0.1796771412454840	0.179677143791334
0.7	0.244755067600357	0.2443036169821510	0.244303617750305
0.8	0.319454500640289	0.3186460093102460	0.318646005190335
0.9	0.403894871267148	0.4025686205525250	0.402568610236483
1	0.497922483110430	0.4959003827831510	0.495900375094189

Example 4 Now consider the nonlinear boundary layer equation

$$2y''' + y''y = 0, 0 \leq x \leq 1 \tag{14}$$

with initial conditions

$$y(0) = 0, y'(0) = 0, y''(0) = B \tag{15}$$

This equation is famously known as the Blasius equation. The aim of solving Blasius equation to get the value $y''(0)$ to evaluate the shear stress at the plate. Blasius equation has been solved using different methods like series expansions, Runge Kutta, differential transformation and others. By solving the Equation (14) with conditions (15) we obtain the vector A for $n = 10$ and $B = 1$

$$A = \begin{bmatrix} -5.258858 \\ 34.500595 \\ -102.128598 \\ 176.867272 \\ -196.065300 \\ 143.930229 \\ -69.967095 \\ 21.716530 \\ -3.904633 \\ 0.309859 \end{bmatrix}$$

Table 6 show that the numerical solutions for non-linear Blasius equation (14) (Example 4) by using presented method ((CP) method) and compared with another numerical methods for $n = 10$ and $B = 1$ (Hb. method is Hybrid block method and Bp is Bernstein polynomials). In all the above the results, it is noticed that the numerical solutions achieved by our method coincide quite well with other methods available in the literature and signify that the proposed method is viable and convergent.

5. Conclusion

In this paper, we introduced an effective operational matrix method for solving nonlinear third-order of non-linear ordinary differential equations by constructing a new matrices using the clique polynomials. The proposed approach has been successfully applied to various numerical examples to demonstrate its applicability and accuracy. Numerical simulations confirm that the approximate solutions are in excellent agreement with solutions obtained by other existing methods or exact solution, and a highly accurate solution can be obtained in a few iterates, which is apparent through numerical results. The proposed algorithm is an efficient and highly promising technique for solving third-order non-linear ordinary differential equations. The method might be applied for a system of differential equations or higher order of boundary value problems.

Acknowledgment

The author would like to thank to express their gratitude to the anonymous reviewers for their constructive feedback and the editor for their thorough evaluation of this manuscript.

Declarations

The author declares that he has no conflicts of interest.

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