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Finite Time Blow Up of Coupled Nonlinear Viscoelastic Wave Equations with Distributed Delay and Strong Damping

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Abstract

In this work, we are concerned with a problem for a coupled non-linear viscoelastic wave equation with distributed delay, strong damping, and source terms. under suitable conditions, we prove the blow up result of solutions.

Keywords

Viscoelastic equation; Blow up; Strong damping; Distributed delay.

2020 Mathematics Subject Classification

Primary 35B40 · Secondary 35L45, 93D15, 93D20 ·

1. Introduction

We consider the following initial-boundary value problem for a coupled system of nonlinear viscoelastic wave equations with distributed delay and strong damping:

$$\begin{cases} u_{tt} - \Delta u - \omega_1 \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ \quad +\mu_1 u_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u_t(x,t-\varrho)d\varrho = f_1(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+ \\ v_{tt} - \Delta v - \omega_2 \Delta v_t + \displaystyle\int_0^t h(t-s)\Delta v(s)ds \\ \quad +\mu_3 v_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v_t(x,t-\varrho)d\varrho = f_2(u,v), \quad (x,t) \in \Omega \times \mathbb{R}_+ \\ u(x,t) = 0, \quad v(x,t) = 0, \quad x \in \partial\Omega \\ u_t(x,-t) = f_0(x,t), \quad v_t(x,-t) = k_0(x,t), \quad (x,t) \in \Omega \times (0,\tau_2) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \\ v(x,0) = v_0(x), v_t,(x,0) = v_1(x), x \in \Omega, \end{cases} \tag{1}$$

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where

$$\begin{cases} f_1(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|u|^p u|v|^{p+2} \\ f_2(u,v) = a_1|u+v|^{2(p+1)}(u+v) + b_1|v|^p v|u|^{p+2}, \end{cases} \tag{2}$$

and $\omega_1, \omega_2, \mu_1, \mu_3, a_1, b_1 > 0$, and τ_1, τ_2 are the time delays with $0 \le \tau_1 < \tau_2$, and μ_2, μ_4 are L^∞ functions, and g, h are differentiable functions.

Viscous materials are the opposite of elastic materials, which possess the ability to store and dissipate mechanical energy. The mechanical properties of these viscous substances are of great importance, as they appear in many applications of the natural sciences. Many authors have paid attention to this problem since the beginning of the new millennium.

In the case of only one equation, and if $w_1 = 0$, that is, for the absence of Δu_t and $\mu_1 = \mu_2 = 0$. Our problem (1) has been studied by [5]. By using the Galerkin method, they established the local existence result. Additionally, they showed that the local solution is global in time under suitable conditions, and with the same rate of decay (polynomial or exponential) of the kernel g. they proved that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution. Additionally, their result has been obtained under weaker conditions than those used by [8].

In [9], the authors consider the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^\gamma u = 0, \tag{3}$$

the authors proved the exponential decay result. This later result has been improved by [5], in which it was shown that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate.

In many works in this field, under assumptions of the kernel g. For the problem (1) and with $\mu_1 \neq 0$, for example in [12], the authors proved a blow up result for the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \end{cases} \tag{4}$$

where g satisfies $\int_0^\infty g(s)ds < (2p-4)/(2p-3)$, the initial data were supported by negative energy like that $\int u_0 u_1 dx > 0$. If $(w > 0)$. In [16], Song et al considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^\infty g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \quad (x,t) \in \Omega \times (0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x). \end{cases} \tag{5}$$

Under suitable assumptions on g, they showed that there were solutions of (5) with initial energy that blow up in finite time. For the same problem (5), in [17], Song et al proved that there were solutions of (5) with positive initial energy that blow up in finite time. In [18], the following problem is studied

$$\begin{cases} u_{tt} - \Delta u - \omega\Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}u_t = |u|^{p-2}u, \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \quad x \in \Omega \\ u(x,t) = 0, \quad x \in \partial\Omega, \end{cases} \tag{6}$$

the author proved the exponential growth result under suitable assumptions.

In [13] the authors considered the following problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds - \varepsilon_1\Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2} \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{7}$$

they showed a blow up result if $p > m$ and established global existence.

In the case of coupled equations, in [1], the authors studied the following system of equations

$$\begin{cases} u_{tt} - \Delta u + u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{8}$$

with nonlinear functions f_1 and f_2 satisfying appropriate conditions. Under certain restrictions imposed on the parameters and the initial data, they obtained numerous results on the existence of weak solutions. They also showed that any weak solution with negative initial energy blows up for a finite period of time by using the same techniques as in [2, 10, 11]. In [4], the authors considered the system:

$$\begin{cases} u_{tt} - \Delta u + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + (a|u|^\theta + b|v|^\vartheta)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{9}$$

they stated and proved that the blows occur in finite time for the solution, under some restrictions on the initial data and (with positive initial energy) for certain conditions on the functions f_1 and f_2. In [15], the authors extended the result of [4] and considered the following nonlinear viscoelastic system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(s)\Delta u(t-s)ds + (a|u|^k + b|v|^l)u_t|u_t|^{m-2} = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^\infty h(s)\Delta v(t-s)ds + (a|u|^\theta + b|v|^\varrho)v_t|v_t|^{r-2} = f_2(u,v), \end{cases} \tag{10}$$

they proved that the solutions of a system of wave equations with a viscoelastic term, degenerate damping, and strong nonlinear sources acting in both equations simultaneously are globally nonexistent, provided that the initial data are sufficiently large in a bounded domain of Ω; see [6, 7, 19].
As a complement to these works, we are working to prove the blow-up result with distributed delay of the problem (1), under appropriate assumptions, and we prove these results using the energy method. In the following, let $c, c_i > 0, i = 1, ..., 12$.
We prove the blow-up result under the following suitable assumptions.
(**A1**) $g, h : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function such that

$$\begin{aligned} g(t) \geq 0 \quad , \quad 1 - \int_0^\infty g(s)\,ds = l_1 > 0, \\ h(t) \geq 0 \quad , \quad 1 - \int_0^\infty h(s)\,ds = l_2 > 0. \end{aligned} \tag{11}$$

(**A2**) There exists a constant $\xi_1, \xi_2 > 0$ such that

$$\begin{aligned} g'(t) \leq -\xi_1 g(t) \quad , \quad t \geq 0, \\ h'(t) \leq -\xi_2 h(t) \quad , \quad t \geq 0. \end{aligned} \tag{12}$$

(**A3**) $\mu_2, \mu_4 : [\tau_1, \tau_2] \to \mathbb{R}$ are L^∞ functions such that

$$\begin{aligned} \left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1 \quad , \quad \delta > \frac{1}{2}, \\ \left(\frac{2\delta - 1}{2}\right) \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho < \mu_3 \quad , \quad \delta > \frac{1}{2}. \end{aligned} \tag{13}$$

2. Blow up in finite time

In this subsection, we prove the blow up result of the solution to problem (1).
First, as in [14], we introduce the new variables

$$\begin{cases} y(x, \rho, \varrho, t) = u_t(x, t - \varrho\rho) \\ z(x, \rho, \varrho, t) = v_t(x, t - \varrho\rho), \end{cases} \tag{14}$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ y(x, 0, \varrho, t) = u_t(x, t), \end{cases} \tag{15}$$

and

$$\begin{cases} \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0 \\ z(x, 0, \varrho, t) = v_t(x, t), \end{cases} \tag{16}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t - s)|u(t) - u(s)|^2 ds dx. \tag{17}$$

Therefore, the problem (1) takes the form

$$\begin{cases} u_{tt} - \Delta u - \omega_1 \Delta u_t + \int_0^t g(t - s)\Delta u(s) ds \\ \quad + \mu_1 u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho = f_1(u, v), \qquad x \in \Omega, t \geq 0 \\ v_{tt} - \Delta v - \omega_2 \Delta v_t + \int_0^t h(t - s)\Delta v(s) ds \\ \quad + \mu_3 v_t + \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x, 1, \varrho, t) d\varrho = f_2(u, v), \qquad x \in \Omega, t \geq 0 \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0 \\ \varrho z_t(x, \rho, \varrho, t) + z_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{18}$$

with initial and boundary conditions

$$\begin{cases} u(x, t) = 0, \quad v(x, t) = 0 \quad x \in \partial\Omega, \\ y(x, \rho, \varrho, 0) = f_0(x, \varrho\rho), \quad z(x, \rho, \varrho, 0) = k_0(x, \varrho\rho) \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x) \\ v(x, 0) = v_0(x), \ v_t(x, 0) = v_1(x), \end{cases} \tag{19}$$

where

$$(x, \rho, \varrho, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Theorem 2.1. *Assume (11),(12), and (13) hold. Let*

$$\begin{cases} -1 < p < \dfrac{4-n}{n-2}, & n \geq 3 \\ p \geq -1, & n = 1,2. \end{cases} \tag{20}$$

Then for any initial data

$$(u_0, u_1, v_0, v_1, f_0, k_0) \in \mathcal{H},$$

where

$$\begin{aligned} \mathcal{H} &= H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)) \\ &\quad \times L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)). \end{aligned}$$

the problem (18) has a unique solution

$$u \in C([0,T]; \mathcal{H}),$$

for some $T > 0$.

Lemma 2.2. *There exists a function* $F(u,v)$ *such that*

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)} \left[u f_1(u,v) + v f_2(u,v) \right] \\ &= \frac{1}{2(\rho+2)} \left[a_1 |u+v|^{2(p+2)} + 2b_1 |uv|^{p+2} \right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \quad \frac{\partial F}{\partial v} = f_2(u,v).$$

we take $a_1 = b_1 = 1$ for convenience.

Lemma 2.3. *[15] There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(\rho+2)} \left(|u|^{2(p+2)} + |v|^{2(p+2)} \right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)} \left(|u|^{2(\rho+2)} + |v|^{2(p+2)} \right). \tag{21}$$

We define the energy functional

Lemma 2.4. *Assume (11),(12),(13), and (20) hold. Let* (u,v,y,z) *be a solution of (18); then* $E(t)$ *is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2} l_1 \|\nabla u\|_2^2 + \frac{1}{2} l_2 \|\nabla v\|_2^2 \\ &\quad + \frac{1}{2}(go\nabla u) + \frac{1}{2}(ho\nabla v) + \frac{1}{2} K(y,z) - \int_\Omega F(u,v)dx, \end{aligned} \tag{22}$$

satisfies

$$\begin{aligned} E'(t) &\leq -c_3 \{ \|u_t\|_2^2 + \|v_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &\quad + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx \} \leq 0, \end{aligned} \tag{23}$$

where

$$K(y,z) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \{ |\mu_2(\varrho)| y^2(x,\rho,\varrho,t) + |\mu_4(\varrho)| z^2(x,\rho,\varrho,t) \} d\varrho d\rho dx. \tag{24}$$

Proof. By multiplying $(18)_1, (18)_2$ by u_t, v_t and integrating over Ω, we get

$$\begin{aligned} \frac{d}{dt} &\quad \{ \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|v_t\|_2^2 + \frac{1}{2} l_1 \|\nabla u\|_2^2 + \frac{1}{2} l_2 \|\nabla v\|_2^2 + \frac{1}{2}(go\nabla u) \\ &\quad + \frac{1}{2}(ho\nabla v) - \int_\Omega F(u,v)dx \} \\ &= -\mu_1 \|u_t\|_2^2 - \int_\Omega u_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx \\ &\quad - \mu_3 \|v_t\|_2^2 - \int_\Omega v_t \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z(x,1,\varrho,t) d\varrho dx \\ &\quad + \frac{1}{2}(g' o\nabla u) - \frac{1}{2} g(t) \|\nabla u\|_2^2 - \omega_1 \|\nabla u_t\|_2^2 \\ &\quad + \frac{1}{2}(h' o\nabla v) - \frac{1}{2} h(t) \|\nabla v\|_2^2 - \omega_2 \|\nabla v_t\|_2^2, \end{aligned} \tag{25}$$

and, from $(18)_3$, $(18)_4$, we have

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ & = -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_2(\varrho)|yy_\rho d\varrho d\rho dx \\ = \quad & +\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,0,\varrho,t)d\varrho dx \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ = \quad & \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)d\varrho)\|u_t\|_2^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx, \end{aligned}$$

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_4(\varrho)|z^2(x,\rho,\varrho,t)d\varrho d\rho dx \\ & = -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}2|\mu_4(\varrho)|zz_\rho d\varrho d\rho dx \\ = \quad & +\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,0,\varrho,t)d\varrho dx \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx \\ = \quad & \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)d\varrho)\|v_t\|_2^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx, \end{aligned}$$

then, we get

$$\begin{aligned} \frac{d}{dt}E(t) \quad = \quad & -\mu_1\|u_t\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|u_t y(x,1,\varrho,t)d\varrho dx + \frac{1}{2}(g' o\nabla u) \\ & -\frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega_1\|\nabla u_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)d\varrho)\|u_t\|_2^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx \\ & -\mu_3\|v_t\|_2^2 - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|v_t z(x,1,\varrho,t)d\varrho dx + \frac{1}{2}(h' o\nabla v) \\ & -\frac{1}{2}h(t)\|\nabla v\|_2^2 - \omega_2\|\nabla v_t\|_2^2 + \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)d\varrho)\|v_t\|_2^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx. \end{aligned} \tag{26}$$

By (25)-(26), we get (22). By using Young's inequality, (11),(12), and (13) in (26), we obtain (23). □

Now we define the functional

$$\begin{aligned} \mathbb{H}(t) = -E(t) \quad = \quad & -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|v_t\|_2^2 - \frac{1}{2}l_1\|\nabla u\|_2^2 - \frac{1}{2}l_2\|\nabla v\|_2^2 \\ & -\frac{1}{2}(go\nabla u) - \frac{1}{2}(ho\nabla v) - \frac{1}{2}K(y,z) \\ & +\frac{1}{2(p+2)}[\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{27}$$

Theorem 2.5. *Assume (11)-(13), and (20) hold. Assume further that $E(0) < 0$; then the solution to problem (18) blows up in finite time.*

Proof. From (22), we have

$$E(t) \le E(0) \le 0. \tag{28}$$

Therefore

$$\begin{aligned} \mathbb{H}'(t) = -E'(t) &\geq c_3(\|u_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &+ \|v_t\|_2^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx), \end{aligned} \tag{29}$$

hence

$$\begin{aligned} \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \geq 0 \\ \mathbb{H}'(t) &\geq c_3 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| z^2(x,1,\varrho,t) d\varrho dx \geq 0, \end{aligned} \tag{30}$$

and

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(p+2)} [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \\ &\leq \frac{c_1}{2(p+2)} [\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \end{aligned} \tag{31}$$

We set

$$\begin{aligned} \mathcal{K}(t) &= \mathbb{H}^{1-\alpha} + \varepsilon \int_\Omega (uu_t + vv_t) dx + \frac{\varepsilon}{2} \int_\Omega (\mu_1 u^2 + \mu_3 v^2) dx \\ &+ \frac{\varepsilon}{2} \int_\Omega (\omega_1 (\nabla u)^2 + \omega_2 (\nabla v)^2) dx, \end{aligned} \tag{32}$$

where $\varepsilon > 0$ will be assigned later and

$$0 < \alpha < \frac{2p+2}{4(p+2)} < 1. \tag{33}$$

By multiplying $(18)_1, (18)_2$ by u, v and taking the derivative of (32), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon(\|u_t\|_2^2 + \|v_t\|_2^2) - \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\ &+ \varepsilon \int_\Omega \nabla u \int_0^t g(t-s) \nabla u(s) ds dx + \varepsilon \int_\Omega \nabla v \int_0^t h(t-s) \nabla v(s) ds dx \\ &- \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t) d\varrho dx \\ &+ \varepsilon [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{34}$$

Using Young's inequality, we obtain

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| u y(x,1,\varrho,t) d\varrho dx &\leq \varepsilon \{\delta_1 (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \|u\|_2^2 \\ &+ \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx\}, \end{aligned} \tag{35}$$

$$\begin{aligned} \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| v z(x,1,\varrho,t) d\varrho dx &\leq \varepsilon \{\delta_2 (\int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| d\varrho) \|v\|_2^2 \\ &+ \frac{1}{4\delta_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_4(\varrho)| z^2(x,1,\varrho,t) d\varrho dx\}, \end{aligned} \tag{36}$$

and, we have

$$\begin{aligned} \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . \nabla u(s) dx ds &= \varepsilon \int_0^t g(t-s) ds \int_\Omega \nabla u . (\nabla u(s) - \nabla u(t)) dx ds \\ &+ \varepsilon \int_0^t g(s) ds \|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2} \int_0^t g(s) ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2} (go\nabla u), \end{aligned} \tag{37}$$

$$\begin{aligned} \varepsilon \int_0^t h(t-s) ds \int_\Omega \nabla v . \nabla v(s) dx ds &= \varepsilon \int_0^t h(t-s) ds \int_\Omega \nabla v . (\nabla v(s) - \nabla v(t)) dx ds \\ &+ \varepsilon \int_0^t h(s) ds \|\nabla v\|_2^2 \\ &\geq \frac{\varepsilon}{2} \int_0^t h(s) ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2} (ho\nabla v), \end{aligned} \tag{38}$$

we obtain, from (34),

$$\begin{aligned}\mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon(\|u_t\|_2^2+\|u_t\|_2^2)\\ &-\varepsilon((1-\frac{1}{2}\int_0^t g(s)ds)\|\nabla u\|_2^2+(1-\frac{1}{2}\int_0^t h(s)ds)\|\nabla v\|_2^2)\\ &-\varepsilon\delta_1(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\varepsilon\delta_2(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2\\ &-\frac{\varepsilon}{2}(go\nabla u)-\frac{\varepsilon}{4\delta_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)d\varrho dx\\ &-\frac{\varepsilon}{2}(ho\nabla v)-\frac{\varepsilon}{4\delta_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|z^2(x,1,\varrho,t)d\varrho dx\\ &+\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].\end{aligned} \tag{39}$$

Therefore, using (30) and by setting δ_1,δ_1 so that, $\frac{1}{4\delta_1 c_3}=\frac{\kappa\mathbb{H}^{-\alpha}(t)}{2}$, and $\frac{1}{4\delta_2 c_3}=\frac{\kappa\mathbb{H}^{-\alpha}(t)}{2}$, substituting in (39), we get

$$\begin{aligned}\mathcal{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon(\|u_t\|_2^2+\|v_t\|_2^2)\\ &-\varepsilon[(1-\frac{1}{2}\int_0^t g(s)ds)]\|\nabla u\|_2^2-\varepsilon[(1-\frac{1}{2}\int_0^t h(s)ds)]\|\nabla v\|_2^2\\ &-\varepsilon\frac{\mathbb{H}^\alpha(t)}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\|u\|_2^2-\frac{\varepsilon}{2}(go\nabla u)\\ &-\varepsilon\frac{\mathbb{H}^\alpha(t)}{2c_3\kappa}(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho)\|v\|_2^2-\frac{\varepsilon}{2}(ho\nabla v)\\ &+\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}].\end{aligned} \tag{40}$$

For $0<a<1$, from (27)

$$\begin{aligned}\varepsilon[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}] &= \varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]\\ &+\varepsilon 2(p+2)(1-a)\mathbb{H}(t)\\ &+\varepsilon(p+2)(1-a)(\|u_t\|_2^2+\|v_t\|_2^2)\\ &+\varepsilon(p+2)(1-a)(1-\int_0^t g(s)ds)\|\nabla u\|_2^2\\ &+\varepsilon(p+2)(1-a)(1-\int_0^t h(s)ds)\|\nabla v\|_2^2\\ &-\varepsilon(p+2)(1-a)(go\nabla u)\\ &-\varepsilon(p+2)(1-a)(ho\nabla v)\\ &+\varepsilon(p+2)(1-a)K(y,z).\end{aligned} \tag{41}$$

Substituting in (40), we get

$$\begin{aligned}\mathcal{K}'(t) &\geq [(1-\alpha)-\varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon[(p+2)(1-a)+1]\Big(\|u_t\|_2^2+\|v_t\|_2^2\Big)\\ &+\varepsilon[(p+2)(1-a)\Big(1-\int_0^t g(s)ds\Big)-\Big(1-\frac{1}{2}\int_0^t g(s)ds\Big)]\|\nabla u\|_2^2\\ &+\varepsilon[(p+2)(1-a)(1-\int_0^t h(s)ds)-\Big(1-\frac{1}{2}\int_0^t h(s)ds\Big)]\|\nabla v\|_2^2\\ &-\varepsilon\frac{\mathbb{H}^\alpha(t)}{2c_3\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\|u\|_2^2-\varepsilon\frac{\mathbb{H}^\alpha(t)}{2c_3\kappa}\Big(\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho\Big)\|v\|_2^2\\ &+\varepsilon(p+2)(1-a)K(y,z)+\varepsilon[(p+2)(1-a)-\frac{1}{2}](go\nabla u+ho\nabla v)\\ &+\varepsilon a[\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}]+\varepsilon 2(p+2)(1-a)\mathbb{H}(t).\end{aligned} \tag{42}$$

Since (20) hold, we obtain by using (31) and (33)

$$\begin{aligned}\mathbb{H}^\alpha(t)\|u\|_2^2 &\leq c_4(\|u\|_{2(p+2)}^{2\alpha(p+2)+2}+\|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2),\\ \mathbb{H}^\alpha(t)\|v\|_2^2 &\leq c_5(\|v\|_{2(p+2)}^{2\alpha(p+2)+2}+\|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2),\end{aligned} \tag{43}$$

for some positive constants c_4,c_5. By using (33) and the algebraic inequality

$$B^\theta\leq(B+1)\leq(1+\frac{1}{b})(B+b),\quad\forall B>0,\quad 0<\theta<1,\quad b>0.$$

We have, $\forall t > 0$

$$\begin{aligned} \|u\|_{2(p+2)}^{2\alpha(p+2)+2} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\ \|v\|_{2(p+2)}^{2\alpha(p+2)+2} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)) \leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \end{aligned} \tag{44}$$

where $d = 1 + \frac{1}{\mathbb{H}(0)}$. Also, since

$$(X+Y)^\gamma \leq C(X^\gamma + Y^\gamma), \quad X, Y > 0, \quad \gamma > 0. \tag{45}$$

We conclude

$$\begin{aligned} \|v\|_{2(p+2)}^{2\alpha(p+2)}\|u\|_2^2 &\leq c_6(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_2^{2(p+2)}) \leq c_7(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}), \\ \|u\|_{2(p+2)}^{2\alpha(p+2)}\|v\|_2^2 &\leq c_8(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_2^{2(p+2)}) \leq c_9(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{46}$$

Substituting (44) and (46) in (43), we get

$$\begin{aligned} \mathbb{H}^\alpha(t)\|u\|_2^2 &\leq c_{10}(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}) + c_{10}\mathbb{H}(t), \\ \mathbb{H}^\alpha(t)\|v\|_2^2 &\leq c_{11}(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) + c_{11}\mathbb{H}(t). \end{aligned} \tag{47}$$

Combining (42) and (47), using (21), we get

$$\begin{aligned} \mathcal{K}'(t) \geq\ & [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon[(p+2)(1-a)+1](\|u_t\|_2^2 + \|v_t\|_2^2) \\ & +\varepsilon\{[(p+2)(1-a)-1] - (\int_0^t g(s)ds)[(p+2)(1-a) - \frac{1}{2}]\}\|\nabla u\|_2^2 \\ & +\varepsilon\{[(p+2)(1-a)-1] - (\int_0^t h(s)ds)[(p+2)(1-a) - \frac{1}{2}]\}\|\nabla v\|_2^2 \\ & +\varepsilon(p+2)(1-a)K(y,z) + \varepsilon[(p+2)(1-a) - \frac{1}{2}](go\nabla u + ho\nabla v) \\ & +\varepsilon(c_0 a - \frac{\lambda_1+\lambda_2}{2c_3\kappa})[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}] \\ & +\varepsilon(2(p+2)(1-a) - \frac{\lambda_1+\lambda_2}{2c_3\kappa})\mathbb{H}(t), \end{aligned} \tag{48}$$

where $\lambda_1 = c_{10}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho$, $\lambda_2 = c_{11}\int_{\tau_1}^{\tau_2}|\mu_4(\varrho)|d\varrho$.
In this stage, we take $a > 0$ small enough so that

$$\alpha_1 = (p+2)(1-a) - 1 > 0,$$

and we assume

$$\max\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\} < \frac{(p+2)(1-a)-1}{((p+2)(1-a) - \frac{1}{2})} = \frac{2\alpha_1}{2\alpha_1+1}, \tag{49}$$

we have

$$\begin{aligned} \alpha_2 &= \{(p+2)(1-a)-1) - \int_0^t g(s)ds((p+2)(1-a) - \frac{1}{2})\} > 0, \\ \alpha_3 &= \{(p+2)(1-a)-1) - \int_0^t h(s)ds((p+2)(1-a) - \frac{1}{2})\} > 0, \end{aligned}$$

then we choose κ so large that

$$\begin{aligned} \alpha_4 &= ac_0 - \frac{\lambda_1+\lambda_2}{2c_3\kappa} > 0, \\ \alpha_5 &= 2(p+2)(1-a) - \frac{\lambda_1+\lambda_2}{2c_3\kappa} > 0. \end{aligned}$$

We fixed κ and a; we appointed ε small enough so that

$$\alpha_6 = (1-\alpha) - \varepsilon\kappa > 0.$$

Thus, for some $\beta > 0$, estimate (48) becomes

$$\begin{aligned} \mathcal{K}'(t) \geq\ & \beta\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & +(go\nabla u) + (ho\nabla v) + K(y,z) \\ & +[\|u\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}]\}. \end{aligned} \tag{50}$$

By (21), for some $\beta_1 > 0$, we obtain

$$\begin{aligned} \mathcal{K}'(t) \geq & \beta_1\{\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ & + (go\nabla u) + (ho\nabla v) + K(y,z) \\ & + [\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]\}, \end{aligned} \tag{51}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{52}$$

Next, using Holder's and Young's inequalities, we have

$$\begin{aligned} |\int_\Omega (uu_t + vv_t)dx|^{\frac{1}{1-\alpha}} \geq & C[\|u\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|u_t\|_2^{\frac{\mu}{1-\alpha}} \\ & + \|v\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|v_t\|_2^{\frac{\mu}{1-\alpha}}], \end{aligned} \tag{53}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.
We take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(p+2).$$

Subsequently, for $s = \frac{2}{(1-2\alpha)}$ and by using (27), we obtain

$$\begin{aligned} \|u\|_{2(p+2)}^{\frac{2}{1-2\alpha}} & \leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\ \|v\|_{2(p+2)}^{\frac{2}{1-2\alpha}} & \leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \quad \forall t \geq 0. \end{aligned}$$

Therefore,

$$\left|\int_\Omega (uu_t + vv_t)dx\right|^{\frac{1}{1-\alpha}} \geq c_{12}[\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2 + \mathbb{H}(t)]. \tag{54}$$

Subsequently,

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) = & (\mathbb{H}^{1-\alpha} + \varepsilon\int_\Omega (uu_t + vv_t)dx + \frac{\varepsilon}{2}\int_\Omega (\mu_1 u^2 + \mu_3 v^2)dx \\ & + \frac{\varepsilon}{2}\int_\Omega (\omega_1 \nabla u^2 + \omega_2 \nabla v^2)dx)^{\frac{1}{1-\alpha}} \\ \leq & c\{\mathbb{H}(t) + |\int_\Omega (uu_t + vv_t)dx|^{\frac{1}{1-\alpha}} + \|u\|_2^{\frac{2}{1-\alpha}} + \|\nabla u\|_2^{\frac{2}{1-\alpha}} \\ & + \|v\|_2^{\frac{2}{1-\alpha}} + \|\nabla v\|_2^{\frac{2}{1-\alpha}}\} \\ \leq & c[\mathbb{H}(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (go\nabla u) \\ & + (ho\nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}]. \end{aligned} \tag{55}$$

From (50) and (55), it gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{56}$$

where $\lambda > 0$, this depends only on β and c.
by integrating (56), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda\frac{\alpha}{(1-\alpha)}t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is complete. □

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Declarations

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