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Asymptotic Analysis of a Dynamic Elasticity System with Nonlinear Source Term and Frictional Boundary Conditions in Thin Domains

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Abstract

In this article, we investigate the asymptotic behavior of solutions to a dynamic linear elasticity problem in a thin domain $\Omega^\varepsilon \subset \mathbb{R}^3$, characterized by a nonlinear source term $\alpha^\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon$ and mixed boundary conditions: a strongly nonlinear friction law on one part of the boundary and a Dirichlet condition on the remainder. Our primary goal is to analyze the limiting behavior of the displacement field u^ε as the thickness parameter ε tends to zero.

Keywords

Asymptotic analysis, Dynamic elasticity system, Frictional boundary conditions, Variational formulation.

2020 Mathematics Subject Classification

Primary 35R35, 76F10 · Secondary 78M35 ·

1. Introduction

The study of the asymptotic behavior of elasticity systems and related models has attracted considerable attention in recent decades due to its fundamental role in understanding physical phenomena across scales. Asymptotic analysis serves as a powerful mathematical framework for examining the limiting behavior of solutions as certain small parameters such as the thickness of a domain or the characteristic scale of material heterogeneities tend to zero. This approach is particularly relevant for the modeling of thin structures (e.g., membranes, plates, and shells), where dimensional reduction transforms complex three-dimensional problems into more tractable lower-dimensional models while preserving essential physical and mechanical properties. Such analyses are of great practical importance in structural mechanics, biomechanics, and materials science, where accurate predictions of deformation, stress distribution, and dynamic response are crucial.

The importance of this line of research is well illustrated by numerous contributions in the literature. For instance, Dilmi [1] investigated the asymptotic behavior of nonlinear viscoelastic problems with Tresca friction law in thin domains, while the authors in [2] analyzed an elasticity system with a nonlinear dissipative term. In a related study, Dilmi [3] examined a quasistatic electro-viscoelastic problem featuring Tresca's friction condition on part of the boundary. The asymptotic justification of elastic plate models was established by Gilbert and Vashakmadze [4], while the corresponding analysis for shells was developed by Chacha and Miloudi [5]. Moreover, Dilmi and Otmani [6] extended the framework to generalized elasticity equations with variable exponents, enriching the understanding of nonlinear effects in complex materials. For further references on partial differential equations formulated in thin domains, the reader may consult, for instance, [7], [8] and [9].

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Beyond these contributions, classical and modern works have laid the theoretical foundations for asymptotic methods in elasticity and related fields. Ciarlet [10] provided a rigorous mathematical theory of thin elastic structures, while Lions [11] developed essential concepts in homogenization theory and variational convergence. Applications of asymptotic and variational techniques span diverse areas, including fluid-structure interactions [12], fracture mechanics [13], and contact problems with nonlinear boundary conditions [14], [15] demonstrating the versatility and broad impact of these analytical tools.

In this paper, we focus on the asymptotic behavior of a dynamic problem for linear elasticity in a thin three-dimensional domain $\Omega^\varepsilon \subset \mathbb{R}^3$, incorporating a nonlinear source term $\alpha^\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon$ and a strongly nonlinear boundary condition of frictional type. Our primary objective is to describe the behavior of the displacement field as the thickness parameter ε to 0. This setting models physically realistic situations where thin elastic structures such as layered composites or membranes are subjected to external loads and frictional constraints on parts of their boundary.

The model presented in this paper introduces several new features compared to previous studies, particularly the work of Dilmi et al. [2]. In addition to the nonlinear source term, this model incorporates a nonlinear exponential-type boundary friction law, which accounts for tangential resistance along the contact surface and leads to a nonlinear variational structure. Moreover, it establishes a coupling between nonlinear internal damping and nonlinear boundary friction within a dynamic elastic system defined in a thin domain with possibly heterogeneous coefficients depending on ε. This configuration, involving mixed boundary conditions (Dirichlet, Neumann, and frictional), allows for a more realistic representation of contact and energy dissipation phenomena in elastic materials, thereby extending the theoretical framework of previous works.

The mathematical analysis relies on a variational formulation that ensures the existence and uniqueness of weak solutions. Through an appropriate scaling transformation to a fixed reference domain, we derive uniform estimates with respect to the small parameter ε, which enable us to prove the convergence of solutions and identify the corresponding limit equations. The resulting reduced system retains the essential nonlinear and dynamic features of the original problem, offering valuable insights into the mechanical response of thin elastic bodies under complex boundary conditions.

The structure of this article is as follows. Section 2 introduces the physical setting and governing equations. Section 3 presents the variational formulation and establishes the existence and uniqueness of solutions. Section 4 describes the scaling to a fixed domain and derives the corresponding a priori estimates. Finally, Section 5 provides the asymptotic analysis, including the convergence results and the characterization of the limit problem.

2. Problem Description

We consider a problem associated with deformations of a homogeneous and isotropic elastic body in a dynamic regime within a thin bounded domain Ω^ε of $\mathbb{R}^3$, with a regular boundary surface Γ^ε partitioned into three measurable parts $\bar{\omega}$, $\bar{\Gamma}_1^\varepsilon$, and $\bar{\Gamma}_L^\varepsilon$, where ω is a bounded domain of $\mathbb{R}^2$ defined by the equation $x_3 = 0$. We assume that ω is the lower boundary of the domain, Γ_L^ε is the lateral boundary, and Γ_1^ε is the upper surface defined by the equation $x_3 = \varepsilon h(x_1, x_2)$, where ε is a small parameter intended to tend to zero $(0 < \varepsilon < 1)$ and $h(\cdot)$ is a function defined on ω such that

$$0 < \underline{h} = h_{\min} \leq h(x_1, x_2) \leq h_{\max} = \bar{h}, \quad \forall (x_1, x_2) \in \omega.$$

We denote $x = (x', x_3) \in \mathbb{R}^3$, $x' = (x_1, x_2) \in \mathbb{R}^2$, so the physical domain Ω^ε is given by

$$\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : x' \in \omega, 0 < x_3 < \varepsilon h(x')\}.$$

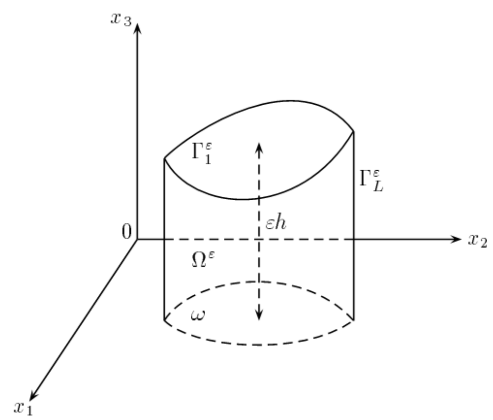

Figure 1: The elastic body Ω^ε.

We denote by $u^\varepsilon(x,t) = (u_1^\varepsilon(x,t), u_2^\varepsilon(x,t), u_3^\varepsilon(x,t))$ the displacement field and $\sigma^\varepsilon(\cdot)$ the stress tensor, whose components $\sigma_{ij}^\varepsilon(\cdot)$, $(1 \leq i,j \leq 3)$ are given by the following law

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda \sum_{k=1}^{3} d_{kk}(u^\varepsilon)\delta_{ij}, \text{ with } d_{ij}(u^\varepsilon) := \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right), 1 \leq i,j \leq 3,$$

where λ, μ are the Lamé coefficients, $d_{ij}(\cdot)$ the strain rate tensor, and δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The equation governing the deformations of a homogeneous and isotropic elastic body with a nonlinear source term in a dynamic regime is as follows

$$\rho \frac{\partial u^\varepsilon}{\partial t} - \text{div}(\sigma^\varepsilon(u^\varepsilon)) + \alpha^\varepsilon |u^\varepsilon|^{p-2} u^\varepsilon = f^\varepsilon, \quad \text{in } \Omega^\varepsilon \times]0, T[, \tag{1}$$

where $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ represents a mass density of external forces, $\rho, \alpha^\varepsilon, p \in \mathbb{R}_+^*$ such that $p \geq 2$.
We now describe the boundary conditions. For this, we define the normal and tangential components of the displacement u^ε by

$$u_n^\varepsilon = u^\varepsilon \cdot n, \quad u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon n,$$

and for the normal and tangential components of the stress tensor σ^ε, the definition is

$$\sigma_n^\varepsilon = (\sigma^\varepsilon \cdot n) \cdot n, \quad \sigma_\tau^\varepsilon = \sigma^\varepsilon \cdot n - (\sigma_n^\varepsilon) n,$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal vector on Γ^ε.

- We assume that the displacement is known on $\Gamma_1^\varepsilon \times]0, T[$ and $\Gamma_L^\varepsilon \times]0, T[$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_1^\varepsilon \times]0, T[, \tag{2}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon \times]0, T[.$$

- On $\omega \times]0, T[$, the displacement is assumed unknown and satisfies the following condition

$$u_n^\varepsilon = 0 \quad \text{on } \omega \times]0, T[. \tag{3}$$

- We also assume the existence of friction on ω, modeled by a power law written as

$$\sigma_\tau^\varepsilon = -k^\varepsilon(x') \left(|u_\tau^\varepsilon|^{q-2} u_\tau^\varepsilon \right), \quad \text{on } \omega \times]0, T[, \tag{4}$$

where $k^\varepsilon : \omega \to \mathbb{R}_+$ is the friction threshold (friction coefficient) and $\frac{1}{p} + \frac{1}{q} = 1$.
The problem (1)-(4) is supplemented by the following initial condition

$$u^\varepsilon(x, 0) = \vartheta_0(x), \quad \forall x \in \Omega^\varepsilon. \tag{5}$$

Remark 2.1. On $\omega \times]0, T[$, the third component of the displacement is zero

$$u_3^\varepsilon = 0, \quad \text{on } \omega \times]0, T[,$$

indeed, according to condition (3), we have

$$u_n^\varepsilon = u_1^\varepsilon n_1 + u_2^\varepsilon n_2 + u_3^\varepsilon n_3 = 0 \quad \text{on } \omega \times]0, T[,$$

where $n = (n_1, n_2, n_3) = (0, 0, -1)$ is the outward unit normal vector to ω. Thus, $u_3^\varepsilon = 0$ on $\omega \times]0, T[$.

3. Variational Formulation

In this section, we derive a variational formulation of the problem (1)-(5). For this, we define the non-empty closed convex subset of $H^1(\Omega^\varepsilon)^3$

$$K^\varepsilon = \{v \in H^1(\Omega^\varepsilon)^3 : v = 0 \text{ on } \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon, \ v_n = 0 \text{ on } \omega\},$$

and to simplify the notation, we denote

$$a(u, v) = 2\mu \sum_{i,j=1}^{3} \int_{\Omega^\varepsilon} d_{ij}(u) d_{ij}(v) dx + \lambda \int_{\Omega^\varepsilon} \text{div}(u) \text{div}(v) dx, \quad \forall u, v \in H^1(\Omega^\varepsilon)^3,$$

and

$$j_q^\varepsilon(u, v) = \int_\omega k^\varepsilon(x') (|u_\tau|^{q-2} u_\tau) \cdot v_\tau dx'.$$

Here, $a(\cdot, \cdot)$ is a bilinear form continuous and coercive on $K^\varepsilon \times K^\varepsilon$.

Lemma 3.1. *If u^ε is a regular solution of the problem (1)-(5), then it satisfies the following variational problem*

$$(PK)^\varepsilon \begin{cases} \text{Find } u^\varepsilon \in K^\varepsilon \text{ such that} \\ \left(\rho \frac{\partial u^\varepsilon}{\partial t}, \varphi \right) + a(u^\varepsilon, \varphi) + \alpha^\varepsilon (|u^\varepsilon|^{p-2} u^\varepsilon, \varphi) + j_q^\varepsilon(u^\varepsilon, \varphi) = (f^\varepsilon, \varphi), \quad \forall \varphi \in K^\varepsilon, \\ u^\varepsilon(x', 0) = \vartheta_0(x'). \end{cases} \tag{6}$$

Proof. Multiplying equation (1) by φ, where $\varphi \in K^\varepsilon$, integrating over Ω^ε, and using Green's formula, we obtain

$$\int_{\Omega^\varepsilon} \rho \frac{\partial u_i^\varepsilon}{\partial t} \varphi_i dx + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial \varphi_i}{\partial x_j} dx - \int_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j \varphi_i dx + \alpha^\varepsilon \int_{\Omega^\varepsilon} |u_i^\varepsilon|^{p-2} u_i^\varepsilon \varphi_i dx = \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i dx. \tag{7}$$

The boundary condition (2) implies that

$$\int_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j \varphi_i dx = \int_\omega \sigma_{ij}^\varepsilon n_j \varphi_i dx'.$$

Moreover, $\sigma_{ij}^\varepsilon n_j = \sigma_{\tau_i}^\varepsilon + \sigma_n^\varepsilon n_j$ and $\varphi_i n_i = 0$ on ω, so

$$\int_{\Gamma^\varepsilon} \sigma_{ij}^\varepsilon n_j \varphi_i dx = \int_\omega \sigma_\tau^\varepsilon \cdot \varphi_\tau dx'.$$

Returning to (7), we get

$$\int_{\Omega^\varepsilon} \rho \frac{\partial u_i^\varepsilon}{\partial t} \varphi_i dx + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial \varphi_i}{\partial x_j} dx - \int_\omega \sigma_\tau^\varepsilon \cdot \varphi_\tau dx' + \int_{\Omega^\varepsilon} k^\varepsilon |u_i^\varepsilon|^{p-2} u_i^\varepsilon \varphi_i dx = \int_{\Omega^\varepsilon} f_i^\varepsilon \varphi_i dx. \tag{8}$$

In (8), we use the fact that

$$\sigma_\tau^\varepsilon = -k^\varepsilon (|u_\tau^\varepsilon|^{q-2} u_\tau^\varepsilon),$$

and

$$\int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial \varphi_i}{\partial x_j} dx = a(u^\varepsilon, \varphi),$$

we obtain the following variational formulation

$$\left(\rho \frac{\partial u^\varepsilon}{\partial t}, \varphi\right) + a(u^\varepsilon, \varphi) + \alpha^\varepsilon (|u^\varepsilon|^{p-2} u^\varepsilon, \varphi) + j_q^\varepsilon(u, \varphi) = (f^\varepsilon, \varphi), \quad \forall \varphi \in K^\varepsilon.$$

□

3.1. Existence and Uniqueness of the Solution

Theorem 3.2. *Under the assumptions*

$$\begin{aligned} f^\varepsilon &\in L^2(0,T; L^2(\Omega^\varepsilon)^3), \\ \vartheta_0 &\in H^1(\Omega^\varepsilon)^3 \cap L^p(\Omega^\varepsilon)^3, \\ k^\varepsilon \in L^\infty(\omega), &\ \text{such that } 0 < k_*^\varepsilon \le k^\varepsilon(x') \le k_{**}^\varepsilon, \end{aligned} \tag{9}$$

there exists a unique solution u^ε *of (6), such that*

$$u^\varepsilon \in L^\infty(0,T; H^1(\Omega^\varepsilon)^3) \cap L^\infty(0,T; L^p(\Omega^\varepsilon)^3),$$

$$\frac{\partial u^\varepsilon}{\partial t} \in L^\infty(0,T; L^2(\Omega^\varepsilon)^3).$$

Proof. **A) Existence.** The proof of Theorem 3.2 will be carried out in three steps.
Step 1: Approximate solution.
We introduce a sequence (w_n^ε) of functions with the following properties:

- ⊛ $\forall j; w_j^\varepsilon \in H^1(\Omega^\varepsilon)^3 \cap L^p(\Omega^\varepsilon)^3$,
- ⊛ The family $\{w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon, \ldots, w_m^\varepsilon\}$ is linearly independent,
- ⊛ The space $K_m^\varepsilon = \text{vect}\{w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon, \ldots, w_m^\varepsilon\}$ generated by the family $\{w_1^\varepsilon, w_2^\varepsilon, \ldots, w_m^\varepsilon\}$ is dense in $K^\varepsilon \cap L^p(\Omega^\varepsilon)^3$.

Let $u_m^\varepsilon = u_m^\varepsilon(t)$ be an approximate solution of problem (1)-(5) such that

$$u_m^\varepsilon(t) = \sum_{j=1}^m \eta_{jm}(t) w_j^\varepsilon, \quad m = 1, 2, 3, \ldots,$$

which satisfies the system of equations

$$\left(\rho \frac{\partial u_m^\varepsilon}{\partial t}, w_j^\varepsilon\right) + a(u_m^\varepsilon, w_j^\varepsilon) + \alpha^\varepsilon (|u_m^\varepsilon|^{p-2} u_m^\varepsilon, w_j^\varepsilon) + j_q(u_m^\varepsilon, w_j^\varepsilon) = (f, w_j^\varepsilon), \quad 1 \le j \le m, \tag{10}$$

which is a nonlinear system of differential equations, supplemented by the initial condition

$$u_m^\varepsilon(x,0) = \vartheta_{0m} = \sum_{i=1}^m X_{jm} w_j^\varepsilon \to \vartheta_0 \text{ as } m \to \infty \text{ in } H^1(\Omega^\varepsilon)^3 \cap L^p(\Omega)^3. \tag{11}$$

Since the family $\{w_1^\varepsilon, w_2^\varepsilon, \ldots, w_m^\varepsilon\}$ is linearly independent, problem (10)-(11) has at least one solution u_m^ε in the interval $[0, t_m]$.

Step 2: A priori estimates.
Multiplying equation (10) by $\eta'_{jm}(t)$ and summing over $j=1$ to m, we obtain

$$\left(\rho\frac{\partial u_m^\varepsilon}{\partial t},\frac{\partial u_m^\varepsilon}{\partial t}\right)+a\left(u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)+\alpha^\varepsilon\left(|u_m^\varepsilon|^{p-2}u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)+j_q\left(u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)=\left(f,\frac{\partial u_m^\varepsilon}{\partial t}\right), \tag{12}$$

Moreover, using the Gâteaux derivative, we have

$$\left(|u_m^\varepsilon|^{p-2}u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)=\frac{1}{p}\frac{d}{dt}\|u_m^\varepsilon(t)\|^p_{L^p(\Omega^\varepsilon)^3}, \tag{13}$$

and

$$j_q\left(u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)=\frac{1}{q}\frac{d}{dt}\|(k^\varepsilon)^{\frac{1}{q}}u_m^\varepsilon(t)\|^q_{L^q(\omega)^2}, \tag{14}$$

we also have

$$a\left(u_m^\varepsilon,\frac{\partial u_m^\varepsilon}{\partial t}\right)=\frac{d}{dt}\left[\mu\|d(u_m^\varepsilon(t))\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{\lambda}{2}\|\mathrm{div}(u_m^\varepsilon(t))\|^2_{L^2(\Omega^\varepsilon)}\right]. \tag{15}$$

Using formulas (13)-(15) in Eq. (12), we obtain

$$\begin{aligned}&\rho\left\|\frac{\partial u_m^\varepsilon(t)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}+\frac{d}{dt}\left[\mu\,\|d(u_m^\varepsilon(t))\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{\lambda}{2}\,\|\mathrm{div}(u_m^\varepsilon(t))\|^2_{L^2(\Omega^\varepsilon)}\right]\\&+\frac{\alpha^\varepsilon}{p}\frac{d}{dt}\|u_m^\varepsilon(t)\|^p_{L^p(\Omega)^3}+\frac{1}{q}\frac{d}{dt}\|(k^\varepsilon)^{\frac{1}{q}}u_m^\varepsilon(t)\|^q_{L^q(\omega)^2}\\&=\left(f,\frac{\partial u_m^\varepsilon(t)}{\partial t}\right).\end{aligned}$$

Integrating the last equation over $]0,t[$ and applying Hölder's and Young's inequalities, we deduce

$$\begin{aligned}&\rho\int_0^t\left\|\frac{\partial u_m^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}ds+\mu\|d(u_m^\varepsilon(t))\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}+\frac{\alpha^\varepsilon}{p}\|u_m^\varepsilon(t)\|^p_{L^p(\Omega^\varepsilon)^3}\\&\le(2\mu+\lambda)\|\vartheta_{0m}\|^2_{H^1(\Omega^\varepsilon)^3}+\frac{\alpha^\varepsilon}{p}\|\vartheta_{0m}\|^p_{L^p(\Omega^\varepsilon)^3}+k^\varepsilon_{**}\|\vartheta_{0m}\|^q_{L^q(\omega)^2}\\&+2\rho\int_0^t\|f(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds+\frac{\rho}{2}\int_0^t\left\|\frac{\partial u_m^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}ds.\end{aligned} \tag{16}$$

Now, using Korn's inequality (26)

$$C_K\|u_m^\varepsilon(t)\|^2_{H^1(\Omega^\varepsilon)^3}\le\|d(u_m^\varepsilon(t))\|^2_{L^2(\Omega)^{3\times3}}.$$

the inequality (16) becomes

$$\begin{aligned}&\frac{\rho}{2}\int_0^t\left\|\frac{\partial u_m^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}ds+\mu C_K\|u_m^\varepsilon(t)\|^2_{H^1(\Omega^\varepsilon)^3}+\frac{\alpha^\varepsilon}{p}\|u_m^\varepsilon(t)\|^p_{L^p(\Omega^\varepsilon)^3}\\&\le2\rho\int_0^t\|f(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds+(2\mu+\lambda)\|\vartheta_{0m}\|^2_{H^1(\Omega^\varepsilon)^3}+\alpha^\varepsilon\|\vartheta_{0m}\|^p_{L^p(\Omega^\varepsilon)^3}+k^\varepsilon_{**}\|\vartheta_{0m}\|^q_{L^q(\omega)^2},\end{aligned}$$

since

$$\begin{aligned}&2\rho\int_0^t\|f(s)\|^2_{L^2(\Omega^\varepsilon)^3}ds+(2\mu+\lambda)\|\vartheta_{0m}\|^2_{H^1(\Omega^\varepsilon)^3}+\alpha^\varepsilon\|\vartheta_{0m}\|^p_{L^p(\Omega^\varepsilon)^3}+k^\varepsilon_{**}\|\vartheta_{0m}\|^q_{L^q(\omega)^2}\\&\le C^\varepsilon,\quad\forall m\in\mathbb{N}^*.\end{aligned}$$

Here, C^ε is a constant independent of m. Thus, we obtain

$$\left\|\frac{\partial u_m^\varepsilon(s)}{\partial t}\right\|^2_{L^\infty(0,T;L^2(\Omega^\varepsilon)^3)}+\|u_m^\varepsilon(t)\|^2_{H^1(\Omega^\varepsilon)^3}+\|u_m^\varepsilon(t)\|^p_{L^p(\Omega^\varepsilon)^3}\le C^\varepsilon. \tag{17}$$

Step 3: Passage to the limit.
From the estimate (17), we conclude

$$u_m^\varepsilon\text{ bounded in }L^\infty(0,T;H^1(\Omega^\varepsilon)^3)\cap L^\infty(0,T;L^p(\Omega^\varepsilon)^3),$$

$$\frac{\partial u_m^\varepsilon}{\partial t}\text{ bounded in }L^2(0,T;L^2(\Omega^\varepsilon)^3),$$

we deduce that we can extract a subsequence u_m^ε such that

$$\begin{aligned}&u_m^\varepsilon\rightharpoonup u\text{ in }L^\infty(0,T;H^1(\Omega^\varepsilon)^3)\cap L^\infty(0,T;L^p(\Omega^\varepsilon)^3),\\&\frac{\partial u_m^\varepsilon}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}\text{ in }L^\infty(0,T;L^2(\Omega^\varepsilon)^3).\end{aligned} \tag{18}$$

The sequences u_m^ε, $\frac{\partial u_m^\varepsilon}{\partial t}$ are bounded in $L^2(0,T;L^2(\Omega^\varepsilon)^3)$, so by Lions' compactness lemma [16], we can deduce

$$u_m^\varepsilon \overset{\text{strongly}}{\longrightarrow} u^\varepsilon \text{ in } L^2(0,T;L^2(\Omega^\varepsilon)^3).$$

Moreover, since $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int_{\Omega^\varepsilon} \left| |u_m^\varepsilon|^{p-2} u_m^\varepsilon \right|^q dx = \int_{\Omega^\varepsilon} |u_m^\varepsilon|^p dx \le C^\varepsilon.$$

Thus, $|u_m^\varepsilon|^{p-2}u_m^\varepsilon$ is bounded in $L^\infty(0,T;L^q(\Omega^\varepsilon)^3)$, which implies that $|u_m^\varepsilon|^{p-2}u_m^\varepsilon \rightharpoonup \chi^\varepsilon$ in $L^\infty(0,T;L^q(\Omega^\varepsilon)^3)$. But since $u_m^\varepsilon \longrightarrow u^\varepsilon$ in $L^2(0,T;L^2(\Omega^\varepsilon)^3)$, we find

$$|u_m^\varepsilon|^{p-2}u_m^\varepsilon \rightharpoonup \chi^\varepsilon = |u^\varepsilon|^{p-2}u^\varepsilon \text{ in } L^\infty(0,T;L^q(\Omega^\varepsilon)^3). \tag{19}$$

Now, let j be fixed, and $l>j$. Then from (10), we have

$$\left(\rho\frac{\partial u_l^\varepsilon}{\partial t}, w_j^\varepsilon\right) + a(u_l^\varepsilon, w_j^\varepsilon) + \alpha^\varepsilon\left(|u_l^\varepsilon|^{p-2}u_l^\varepsilon, w_j^\varepsilon\right) + j_q(u_l^\varepsilon, w_j^\varepsilon) = (f, w_j^\varepsilon), \quad 1\le j\le l. \tag{20}$$

Moreover, from (18) and (19), it follows that

$$\left(|u_l^\varepsilon|^{p-2}u_l^\varepsilon, w_j^\varepsilon\right) \rightharpoonup \left(|u^\varepsilon|^{p-2}u^\varepsilon, w_j^\varepsilon\right) \text{ in } L^\infty(0,T),$$

$$\left(\frac{\partial u_l^\varepsilon}{\partial t}, w_j^\varepsilon\right) \rightharpoonup \left(\frac{\partial u^\varepsilon}{\partial t}, w_j^\varepsilon\right) \text{ in } L^\infty(0,T),$$

thus

$$a(u_l^\varepsilon, w_j^\varepsilon) \rightharpoonup a(u^\varepsilon, w_j^\varepsilon) \text{ in } L^\infty(0,T).$$

Consequently, when $l\longrightarrow\infty$, formula (20) becomes

$$\left(\rho\frac{\partial u^\varepsilon}{\partial t}, w_j^\varepsilon\right) + a(u^\varepsilon, w_j^\varepsilon) + \alpha^\varepsilon\left(|u^\varepsilon|^{p-2}u^\varepsilon, w_j^\varepsilon\right) + j_q(u^\varepsilon, w_j^\varepsilon) = (f^\varepsilon, w_j^\varepsilon),$$

for all $w_j^\varepsilon \in K_m^\varepsilon$ and all $1\le j\le m$.
By the density of K_m^ε in $K^\varepsilon \cap L^p(\Omega^\varepsilon)^3$, we conclude that

$$\left(\rho\frac{\partial u^\varepsilon}{\partial t}, \varphi\right) + a(u^\varepsilon, \varphi) + \alpha^\varepsilon\left(|u^\varepsilon|^{p-2}u^\varepsilon, \varphi\right) + j_q(u^\varepsilon, \varphi) = (f^\varepsilon, \varphi), \quad \forall\varphi\in K^\varepsilon. \tag{21}$$

Thus, u^ε satisfies (1)-(3).
B) Uniqueness of the solution.
Let $u^{\varepsilon,1}$ and $u^{\varepsilon,2}$ be two solutions of problem (6). We have

$$\left(\rho\frac{\partial u^{\varepsilon,1}}{\partial t}, \varphi\right) + a(u^{\varepsilon,1}, \varphi) + \alpha^\varepsilon\left(|u^{\varepsilon,1}|^{p-2}u^{\varepsilon,1}, \varphi\right) + j_q^\varepsilon(u^{\varepsilon,1}, \varphi) = (f^\varepsilon, \varphi), \quad \forall\varphi\in K^\varepsilon, \tag{22}$$

and

$$\left(\rho\frac{\partial u^{\varepsilon,2}}{\partial t}, \varphi\right) + a(u^{\varepsilon,2}, \varphi) + \alpha^\varepsilon\left(|u^{\varepsilon,2}|^{p-2}u^{\varepsilon,2}, \varphi\right) + j_q^\varepsilon(u^{\varepsilon,2}, \varphi) = (f^\varepsilon, \varphi), \quad \forall\varphi\in K^\varepsilon. \tag{23}$$

Choosing $\varphi = u^{\varepsilon,2}-u^{\varepsilon,1}$ in (22) and $\varphi = u^{\varepsilon,1}-u^{\varepsilon,2}$ in (23), then summing the two equations, we obtain

$$\begin{aligned}
&\rho\left(\frac{\partial(u^{\varepsilon,1}-u^{\varepsilon,2})}{\partial t}, u^{\varepsilon,1}-u^{\varepsilon,2}\right) + a\left(u^{\varepsilon,1}-u^{\varepsilon,2}, u^{\varepsilon,1}-u^{\varepsilon,2}\right)\\
&+\alpha^\varepsilon\left(|u^{\varepsilon,1}|^{p-2}u^{\varepsilon,1} - |u^{\varepsilon,2}|^{p-2}u^{\varepsilon,2}, u^{\varepsilon,1}-u^{\varepsilon,2}\right)\\
&+j_q^\varepsilon\left(u^{\varepsilon,1}, u^{\varepsilon,1}-u^{\varepsilon,2}\right) - j_q^\varepsilon\left(u^{\varepsilon,2}, u^{\varepsilon,1}-u^{\varepsilon,2}\right)\\
&=0.
\end{aligned}$$

Using the fact that

$$\left(|u^{\varepsilon,1}|^{p-2}u^{\varepsilon,1} - |u^{\varepsilon,2}|^{p-2}u^{\varepsilon,2}, u^{\varepsilon,1}-u^{\varepsilon,2}\right) \ge 0,$$

and

$$j_q^\varepsilon\left(u^{\varepsilon,1}, u^{\varepsilon,1}-u^{\varepsilon,2}\right) - j_q^\varepsilon\left(u^{\varepsilon,2}, u^{\varepsilon,1}-u^{\varepsilon,2}\right) \ge 0,$$

we conclude that

$$\rho\frac{d}{dt}\left\|u^{\varepsilon,1}(t)-u^{\varepsilon,2}(t)\right\|^2_{L^2(\Omega^\varepsilon)^3} + 2\mu C_K\left\|u^{\varepsilon,1}-u^{\varepsilon,2}\right\|^2_{H^1(\Omega^\varepsilon)^3} \le 0. \tag{24}$$

Now, integrating (24) over $]0,t[$, we obtain

$$\left\|u^{\varepsilon,1}-u^{\varepsilon,2}\right\|^2_{L^2(\Omega^\varepsilon)^3} + \int_0^t \left\|u^{\varepsilon,1}(s)-u^{\varepsilon,2}(s)\right\|^2_{H^1(\Omega^\varepsilon)^3} ds \le 0.$$

Hence $u^{\varepsilon,1}=u^{\varepsilon,2}$, which completes the proof. □

4. Change of Domain and A Priori Estimates

Our domain Ω^ε varies with ε, so to study the asymptotic analysis of the problem, we will first transform the domain Ω^ε into a fixed domain Ω. For this, we will use the scaling technique

$$z = \frac{x_3}{\varepsilon}.$$

Thus, the fixed domain Ω is defined by

$$\Omega = \{(x', z) \in \mathbb{R}^3 : x' \in \omega, 0 < z < h(x')\}.$$

We denote its boundary by $\Gamma = \overline{\omega} \cup \overline{\Gamma_L} \cup \overline{\Gamma_1}$.
We define new unknowns on Ω

$$\begin{cases} \hat{u}_1^\varepsilon(x', z, t) = u_1^\varepsilon(x', x_3, t), \\ \hat{u}_2^\varepsilon(x', z, t) = u_2^\varepsilon(x', x_3, t), \\ \hat{u}_3^\varepsilon(x', z, t) = u_3^\varepsilon(x, x_3, t). \end{cases}$$

For the problem data, we assume they depend on ε in the following way

$$\begin{cases} \hat{f}_i(x', z, t) = f_i^\varepsilon(x, x_3, t), \quad i = 1, 2, 3, \\ \hat{k} = \varepsilon k^\varepsilon, \quad \hat{\alpha} = \varepsilon^2 \alpha^\varepsilon, \end{cases}$$

where $\hat{f}_i$ $(i = 1, 2, 3)$, $\hat{k}$, and $\hat{\alpha}$ are independent of ε.
Moreover, we define function spaces on Ω

$$K = \{\varphi \in H^1(\Omega)^3 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L \text{ and } \varphi \cdot n = 0 \text{ on } \omega\},$$

$$\Pi(K) = \{\varphi \in H^1(\Omega)^2 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L\},$$

$$V_z = \left\{ v \in L^2(\Omega)^2 : \frac{\partial v}{\partial z} \in L^2(\Omega)^2, \text{ and } v = 0 \text{ on } \Gamma_1 \right\},$$

V_z is a Banach space equipped with the norm

$$\|v\|_{V_z} = \left(\|v\|^2_{L^2(\Omega)^2} + \left\| \frac{\partial v}{\partial z} \right\|^2_{L^2(\Omega)^2} \right)^{\frac{1}{2}}.$$

By multiplying (6) by ε and transforming to the fixed domain Ω, we show that the problem (PK^ε) is equivalent to the problem (PK) given by

$$(PK) \begin{cases} \text{Find } \hat{u}^\varepsilon \in K \text{ such that} \\ \varepsilon^2 \rho \sum_{i=1}^{2} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}, \hat{\varphi}_i \right) + \varepsilon^4 \rho \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}, \hat{\varphi}_3 \right) + \hat{a}(\hat{u}^\varepsilon, \hat{\varphi}) + \hat{\alpha} \sum_{i=1}^{2} \left(|\hat{u}_i^\varepsilon|^{p-2} \hat{u}_i^\varepsilon, \hat{\varphi}_i \right) \\ + \hat{\alpha}\varepsilon^2 (|\hat{u}_3^\varepsilon|^{p-2} \hat{u}_3^\varepsilon, \hat{\varphi}_3) + \hat{J}_q(\hat{u}^\varepsilon, \hat{\varphi}) \\ = \sum_{i=1}^{3} (\hat{f}_i, \hat{\varphi}_i) + \varepsilon(\hat{f}_3, \hat{\varphi}_3), \quad \forall \hat{\varphi} \in K, \\ \hat{u}^\varepsilon(0) = \hat{\vartheta}_0, \end{cases} \tag{25}$$

where

$$\hat{J}_q(\hat{u}^\varepsilon, \hat{\varphi}) = \sum_{i=1}^{2} \int_\omega \hat{k} |\hat{u}^\varepsilon|^{q-2} \hat{u}_i^\varepsilon \hat{\varphi}_i \, dx',$$

and

$$\hat{a}(\hat{u}^\varepsilon, \hat{\varphi}) = \mu\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial \hat{\varphi}_i}{\partial x_j} \, dx' dz$$

$$+ \mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \left(\frac{\partial \hat{\varphi}_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{\varphi}_3}{\partial x_i} \right) dx' dz$$

$$+ 2\mu\varepsilon^2 \int_\Omega \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \frac{\partial \hat{\varphi}_3}{\partial z} \, dx'^2 \int_\Omega \operatorname{div}(\hat{u}^\varepsilon) \operatorname{div}(\hat{\varphi}) \, dx dz.$$

We now attempt to derive a priori estimates for $\hat{u}^\varepsilon$. For this, we need the following lemmas.

Lemma 4.1. ***(Poincaré Inequality [17])*** *Recall that $0 < h(x') < \bar{h}, \forall x' \in \omega$. We have the following inequality*

$$\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)} \leq \bar{h} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}, \quad i = 1, 2.$$

Lemma 4.2. ***(Korn's Inequality [18])*** *For all $\varphi \in K^\varepsilon$, we have*

$$C_K \|\nabla \varphi\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} \leq \|d(\varphi)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}, \tag{26}$$

where C_K is a positive constant independent of ε and φ.

Theorem 4.3. *Under the following hypotheses*

$$f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t} \in L^2(0,T,L^2(\Omega^\varepsilon)^3), \quad \vartheta_0 = 0, \ f^\varepsilon(0) = 0,$$

there exists a constant c independent of ε such that

$$\begin{aligned}&\sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)} + \|\hat{u}_i^\varepsilon\|^p_{L^p(\Omega)}\right) \\ &+ \sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \|\varepsilon\hat{u}_3^\varepsilon\|^p_{L^p(\Omega)} \\ &\leq c,\end{aligned} \tag{27}$$

$$\begin{aligned}&\sum_{i=1}^{2}\left(\left\|\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2\frac{\partial}{\partial x_i}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right)\right\|^2_{L^2(\Omega)}\right) \\ &+ \sum_{i,j=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left\|\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \left\|\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} \\ &\leq c.\end{aligned} \tag{28}$$

Proof. Let u^ε be the solution of problem (6). Taking $\varphi = \frac{\partial u^\varepsilon}{\partial t}$, we have

$$\rho\left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right) + a\left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right) + \alpha^\varepsilon\left(|u^\varepsilon|^{p-2}u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right) + j_q^\varepsilon\left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right) = \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right),$$

hence

$$\rho\left\|\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{1}{2}\frac{d}{dt}\left[a(u^\varepsilon,u^\varepsilon) + \frac{\alpha^\varepsilon}{p}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} + \frac{1}{q}\|(k^\varepsilon)^{\frac{1}{q}}u^\varepsilon\|^q_{L^q(\omega)^3}\right] = \left(f^\varepsilon, \frac{\partial u^\varepsilon}{\partial t}\right).$$

Integrating over $]0,t[$, we obtain

$$\begin{aligned}&\rho\int_0^t\left\|\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} ds + a(u^\varepsilon,u^\varepsilon) + \frac{\alpha^\varepsilon}{p}\|\vartheta_0\|^p_{L^p(\Omega)^3} + \frac{k_*^\varepsilon}{q}\|u^\varepsilon\|^q_{L^q(\omega)^2} \\ &\leq 2\int_0^t\left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds,\end{aligned}$$

since

$$2\int_0^t\left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds = 2(f^\varepsilon,u^\varepsilon) - 2(f^\varepsilon(0),\vartheta_0) - 2\int_0^t\left(\frac{\partial f^\varepsilon(s)}{\partial t}, u^\varepsilon(s)\right) ds,$$

using the Poincaré inequality

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq \varepsilon\bar{h}\|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}},$$

and Young's inequality

$$ab \leq \eta^2\frac{a^2}{2} + \eta^{-2}\frac{b^2}{2}, \quad \forall(a,b)\in\mathbb{R}^2, \forall\eta,$$

we obtain

$$\begin{aligned}\left|2\int_0^t\left(f^\varepsilon(s), \frac{\partial u^\varepsilon(s)}{\partial t}\right) ds\right| \leq\ & \mu C_K\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \mu C_K\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}ds \\ &+\frac{4\varepsilon^2\bar{h}^2}{\mu C_K}\|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} + \frac{4(\varepsilon\bar{h})^2}{\mu C_K}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega)^3} ds.\end{aligned} \tag{29}$$

From (29) and Korn's inequality (26), we deduce the existence of a constant $C_K > 0$ independent of ε such that

$$\begin{aligned}&\mu C_K\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \frac{\alpha^\varepsilon}{p}\|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3} \\ &\leq \mu C_K\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} ds + \frac{4\varepsilon^2\bar{h}^2}{\mu C_K}\|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^2} + \frac{4(\varepsilon\bar{h}^2)}{\mu C_K}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} ds.\end{aligned} \tag{30}$$

Multiplying (30) by ε and using the equality

$$\varepsilon^2\|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^3} = \varepsilon^{-1}\|\hat{f}\|^2_{L^2(\Omega)^3},$$

we obtain

$$\mu C_K\varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} + \frac{\alpha^\varepsilon}{p}\varepsilon\|u^\varepsilon(s)\|^p_{L^p(\Omega^\varepsilon)^3} \leq \mu C_K\int_0^t\varepsilon\|\nabla u^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon)^{3\times3}} ds + A,$$

where A is a constant independent of ε, with

$$A = \frac{4\hat{h}^2}{\mu C_K}\|\hat{f}\|^2_{L^\infty(0,T,L^2(\Omega)^3)} + \frac{4\bar{h}^2}{\mu C_k}\left\|\frac{\partial \hat{f}}{\partial t}\right\|^2_{L^2(0,T,L^2(\Omega)^3)}.$$

Using Gronwall's lemma, we find

$$\frac{1}{\varepsilon}\|u^\varepsilon\|^p_{L^p(\Omega^\varepsilon)^3} + \varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq c.$$

Thus, the estimate (27) is proved.
To show the a priori estimate (28), we differentiate (21) with respect to t, then take $\varphi = \frac{\partial u^\varepsilon}{\partial t}$, yielding

$$\begin{aligned}&\left(\rho\frac{\partial^2 u^\varepsilon}{\partial t^2}, \frac{\partial u^\varepsilon}{\partial t}\right) + a\left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right) + (p-1)\alpha^\varepsilon\left(|u^\varepsilon|^{p-2}\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right)\\ &+ (q-1)\int_\omega k^\varepsilon|u^\varepsilon|^{q-2}\frac{\partial u^\varepsilon}{\partial t}\cdot\frac{\partial u^\varepsilon}{\partial t}\,dx'\\ &= \left(\frac{\partial f^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right),\end{aligned}$$

since $\int_\omega k^\varepsilon|u^\varepsilon|^{q-2}\frac{\partial u^\varepsilon}{\partial t}\cdot\frac{\partial u^\varepsilon}{\partial t}\,dx' \geq 0$ and $\left(|u^\varepsilon|^{p-2}\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right) \geq 0$, we obtain

$$\frac{\rho}{2}\frac{d}{dt}\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} + a\left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial u^\varepsilon}{\partial t}\right) \leq \left(\frac{\partial u^\varepsilon}{\partial t}, \frac{\partial f^\varepsilon}{\partial t}\right).$$

Integrating over $]0,t[$ and using Korn's inequality (26), we find

$$\rho\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} + 2\mu C_K\int_0^t\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} = 2\int_0^t\left(\frac{\partial f^\varepsilon(s)}{\partial t}, \frac{\partial u^\varepsilon(s)}{\partial t}\right)ds,$$

from which we deduce

$$\begin{aligned}&\rho\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} + 2\mu C_K\int_0^t\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\\ &\leq \mu C_K\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}} + \mu C_K\int_0^t\left\|\nabla\frac{\partial u^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds.\\ &+ \frac{4(\varepsilon\bar{h})^2}{\mu C_K}\int_0^t\left\|\frac{\partial f^\varepsilon(s)}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3}ds\end{aligned} \tag{31}$$

Now, multiplying (31) by ε, we obtain

$$\rho\varepsilon\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^3} + \mu C_K\int_0^t\varepsilon\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}ds \leq B,$$

where B is a constant independent of ε

$$B = \frac{4\bar{h}^2}{\mu C_K}\left\|\frac{\partial \hat{f}}{\partial t}\right\|^2_{L^2(0,T,L^2(\Omega)^3)}.$$

Thus, the estimate (28) is proved. □

5. Convergence Results and Limit Problem

We have established a priori estimates for the solution of problem (25) in the previous section. The natural question is: what is the asymptotic behavior of the elastic membrane when the thickness becomes very small? Mathematically, this amounts to determining whether the displacement field $\hat{u}^\varepsilon$ has a limit as ε tends to zero and what problem this limit satisfies. In this section, we will attempt to answer these questions.

Lemma 5.1. *Under the hypotheses of Theorem 4.3, there exists $u_i^* \in L^2(0,T,V_z) \cap L^2(0,T,L^p(\Omega)), i = 1,2$ such that for any subsequence of $\hat{u}^\varepsilon$, still denoted $\hat{u}^\varepsilon$, we have the following convergence results:*

$$\hat{u}^\varepsilon \rightharpoonup u_i^*, i = 1,2, \ \ in\ L^2(0,T,V_z) \cap L^2(0,T,L^p(\Omega)), \tag{32}$$

$$\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_i^*}{\partial t}, i = 1,2, \ \ in\ L^2(0,T,V_z),$$

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup 0; \varepsilon\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial x_j\partial t} \rightharpoonup 0, i,j = 1,2, \ \ in\ L^2(0,T,L^2(\Omega)), \tag{33}$$

$$\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup 0, i = 1,2, \ \ in\ L^2(0,T,L^2(\Omega)), \tag{34}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0, i = 1, 2, \ \ in\ L^2(0, T, L^2(\Omega)), \tag{35}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0, \ \ in\ L^2(0, T, L^2(\Omega)), \tag{36}$$

$$\hat{u}_i^\varepsilon \longrightarrow u_i^*, i = 1, 2, \ \ in\ L^2(0, T, L^2(\Omega)), \tag{37}$$

$$|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon \rightharpoonup |u_i^*|^{p-2}u_i^*, i = 1, 2\ \ in\ L^2(0, T, L^q(\Omega)). \tag{38}$$

Proof. According to Theorem 4.3, there exists a constant c independent of ε such that

$$\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2 \leq c, \quad i = 1, 2.$$

Using this estimate and Poincaré's inequality

$$\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)}^2 \leq \bar{h} \left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Omega)}^2, \quad i = 1, 2,$$

we deduce that the sequence $\hat{u}_i^\varepsilon$ is bounded in $L^2(0, T, V_z) \cap L^2(0, T, L^p(\Omega))$, hence the weak convergence result. Similarly, from (28) and Poincaré's inequality for $\frac{\partial \hat{u}_i^\varepsilon}{\partial t}$, we deduce that $\frac{\partial \hat{u}_i^\varepsilon}{\partial t}$ is bounded in $L^2(0, T, V_z)$ and consequently converges to a limit v, and since $\hat{u}_i^\varepsilon \rightharpoonup u_i^*$, therefore $v = \frac{\partial u_i^*}{\partial t}$. For (33)-(36), according to (27), (28) and (32).

The sequences $\hat{u}_i^\varepsilon$, $\frac{\partial \hat{u}_i^\varepsilon}{\partial t}$, $i = 1, 2$ are bounded in $L^2(0, T, V_z)$, so by Lions' compactness lemma [11], we can deduce

$$\hat{u}_i^\varepsilon \longrightarrow u_i^* \text{ strongly in } L^2(0, T; L^2(\Omega)), \quad i = 1, 2.$$

On the other hand, we have

$$\int_{\Omega^\varepsilon} \left| |\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon \right|^q dx = \int_{\Omega^\varepsilon} |\hat{u}_i^\varepsilon|^p dx \leq C, \quad i = 1, 2.$$

Therefore, $|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon$ is bounded in $L^2(0, T; L^q(\Omega^\varepsilon))$. Since $\hat{u}_i^\varepsilon \overset{\text{strongly}}{\longrightarrow} u_i^*$ in $L^2(0, T; L^2(\Omega))$, hence (38). □

Theorem 5.2. *Under the assumptions of Theorem 4.3, the limit (u_1^*, u_2^*) satisfies the variational formulation*

$$\mu \sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^*}{\partial z} \frac{\partial \hat{\varphi}_i}{\partial z} dxdz + \hat{\alpha} \sum_{i=1}^{2} \int_\Omega |u_i^*|^{p-2} u_i^* \hat{\varphi}_i dxdz + \hat{J}_q(u^*, \hat{\varphi}) \tag{39}$$
$$= \sum_{i=1}^{2} \int_\Omega \hat{f}_i \hat{\varphi}_i dx, \quad \forall \hat{\varphi} \in \Pi(K),$$

also (u_1^*, u_2^*) satisfies the following problem

$$-\mu \frac{\partial^2 u_i^*}{\partial z^2}(t) + \hat{\alpha} |u_i^*|^{p-2} u_i^*(t) = \hat{f}_i(t), \quad i = 1, 2\ \ in\ L^2(\Omega), \tag{40}$$

Proof. Recall that the variational formulation (25) can be written as

$$\varepsilon^2 \sum_{i=1}^{3} \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}, \hat{\varphi}_i \right) + \varepsilon^4 \sum_{i=1}^{3} \left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}, \hat{\varphi}_3 \right) + \hat{a}\left(\hat{u}^\varepsilon, \hat{\varphi}\right) + \hat{\alpha} \sum_{i=1}^{2} \left(|\hat{u}_i^\varepsilon|^{p-2}\hat{u}_i^\varepsilon, \hat{\varphi}_i \right)$$
$$+ \hat{\alpha}\varepsilon^2 \left(|\hat{u}_3^\varepsilon|^{p-2}\hat{u}_3^\varepsilon, \hat{\varphi}_3 \right) + \hat{J}_q(\hat{u}^\varepsilon, \hat{\varphi})$$
$$= \sum_{i=1}^{3} \left(\hat{f}_i, \hat{\varphi}_i \right) + \varepsilon \left(\hat{f}_3, \hat{\varphi}_3 \right), \quad \forall \hat{\varphi} \in K.$$

Letting ε tend to zero, and using the convergence results from Lemma 5.1, we obtain

$$\mu \sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^*}{\partial z} \cdot \frac{\partial \hat{\varphi}_i}{\partial z} dx'dz + \hat{\alpha} \int_\Omega |u_i^*|^{p-2} u_i^*, \hat{\varphi}_i dx'dz + \hat{J}_q(u^*, \hat{\varphi}) = \sum_{i=1}^{2} \int_\Omega \hat{f}_i \hat{\varphi}_i dx'dz.$$

We now choose in this variational formulation

$$\hat{\varphi}_i \in H_0^1(\Omega), \quad i = 1, 2,$$

therefore, we find

$$\mu \sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^*}{\partial z} \frac{\partial \hat{\varphi}_i}{\partial z} dxdz + \hat{\alpha} \sum_{i=1}^{2} \int_\Omega |u_i^*|^{p-2} u_i^* \cdot \hat{\varphi}_i dxdz = \sum_{i=1}^{2} \int_\Omega \hat{f}_i \hat{\varphi}_i dxdz.$$

Using Green's formula, then by choosing $\hat{\varphi}_1 \in H_0^1(\Omega)$ and $\hat{\varphi}_2 = 0$, then $\hat{\varphi}_2 \in H_0^1(\Omega)$ and $\hat{\varphi}_1 = 0$, we obtain

$$-\int_\Omega \mu \frac{\partial}{\partial z} \left(\frac{\partial u_i^*}{\partial z} \right) \hat{\varphi}_i dx'dz + \hat{\alpha} \int_\Omega |u_i^*|^{p-2} u_i^* \cdot \hat{\varphi}_i dx'dz + \int_\Omega \hat{f}_i \hat{\varphi}_i dx'dz, \quad \forall \hat{\varphi}_i \in H_0^1(\Omega), \quad i = 1, 2.$$

Consequently

$$-\mu \frac{\partial^2 u_i^*(t)}{\partial z^2} + \hat{\alpha} |u_i^*(t)|^{p-2} u_i^*(t) = \hat{f}_i(t), \quad i = 1, 2 \ \text{ in } \ H^{-1}(\Omega), \quad \forall t \in]0, T[, \tag{41}$$

and since $\hat{f}_i \in L^2(\Omega)$, $\quad i = 1, 2$ then (40) holds in $L^2(\Omega)$). □

Theorem 5.3. *Let*

$$\tau^*(x,t) = \frac{\partial u^*}{\partial z}(x,0,t) \quad \textit{and} \quad s^*(x,t) = u^*(x,0,t),$$

the traces of the displacement $u^* = (u_1^*, u_2^*)$ *on* ω. *Under the same assumptions as Theorem 5.2,* τ^* *and* s^* *satisfy the following inequality*

$$\int_\omega \left(\hat{k}|s^*|^{q-2}s^* - \mu\tau^*\right)\hat{\varphi}dx' \geq 0, \quad \forall \hat{\varphi} \in L^2(\omega)^2, \tag{42}$$

and the friction condition of the power law

$$\mu\tau^* = \hat{k}|s^*|^{q-2}s^*, \textit{ a.e. in } \omega \times]0,T[,$$

also (u^*, s^*) *satisfy the weak formulation*

$$\int_\omega \left(\tilde{F} - \frac{h}{2}s^* + \int_0^h u^*(x,z,t)dz + \tilde{U}\right)\nabla\hat{\varphi}(x')dx' = 0, \quad \forall \hat{\varphi} \in H^1(\omega), \tag{43}$$

where

$$\tilde{F}(x,h,t) = \frac{1}{\mu}\int_0^h F(x,h,t)dz - \frac{h}{2\mu}F(x,h,t), \quad \tilde{U}(x,h,t) = -\frac{\hat{\alpha}}{\mu}\int_0^h U(x,z,t)dz + \frac{\hat{\alpha}h}{2\mu}U(x,h,t),$$

$$F(x,z,t) = \int_0^z\int_0^\zeta \hat{f}(x,\eta,t)d\eta d\zeta, \; U(x,z,t) = \int_0^z\int_0^\zeta |u^*|^{p-2}u^*(x,\eta,t)d\eta d\zeta.$$

Proof. From the variational formulation (39), we have for all $\hat{\varphi} \in \Pi(K)$

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial\hat{\varphi}_i}{\partial z}dx'dz + \hat{\alpha}\sum_{i=1}^2\int_\Omega |u_i^*|^{p-2}u_i^*\hat{\varphi}_i dx'dz + \hat{J}_q(u^*,\hat{\varphi})$$
$$= \sum_{i=1}^2\int_\Omega \hat{f}_i\hat{\varphi}_i dx'dz,$$

and since $n = (0,0,-1)$ on ω, using Green's formula, it follows that

$$-\mu\sum_{i=1}^2\int_\Omega \frac{\partial^2 u_i^*}{\partial z^2}\hat{\varphi}_i dx'dz + \hat{\alpha}\sum_{i=1}^2\int_\Omega |u_i^*|^{p-2}u_i^*\hat{\varphi}_i dxdz - \int_\omega \mu\tau^*\hat{\varphi}_i dx' + \int_\omega \hat{k}|s^*|^{q-2}s^*\hat{\varphi}_\tau dx'$$
$$= \sum_{i=1}^2\int_\Omega \hat{f}_i\hat{\varphi}dx'dz.$$

From (40), we obtain

$$-\int_\omega \mu\tau^*\hat{\varphi}dx' + \int_\omega \hat{k}|s^*|^{q-2}s^*\hat{\varphi}dx' = 0, \; \hat{\varphi} \in \mathcal{D}(\omega)^2. \tag{44}$$

Therefore, by the density of $\mathcal{D}(\omega)^2$ in $L^2(\omega)^2$ we deduce (42). From (44), we obtain

$$\mu\tau^* = \hat{k}|s^*|^{q-2}s^*, \text{ a.e. on } \omega \times]0,T[.$$

To prove (43), by integrating equation (40) twice from 0 to z, we find

$$u^*(x',\eta,t) = s_i^* + z\tau_i^* + \frac{\hat{\alpha}}{\mu}\int_0^z\int_0^\zeta |u_i^*|^{p-2}u_i^*(x',\eta,t)d\eta d\zeta - \frac{1}{\mu}\int_0^z\int_0^\zeta \hat{f}_i(x',\eta,t)d\eta d\zeta, \tag{45}$$

in particular for $z = h(x)$, thus we have

$$s_i^* + h\tau_i^* = -\frac{\alpha}{\mu}\int_0^h\int_0^\zeta |u_i^*|^{p-2}u^*(x,\eta,t)d\eta d\zeta + \frac{1}{\mu}\int_0^h\int_0^\zeta f_i(x,\eta,t)d\eta d\zeta. \tag{46}$$

Integrating (45) between 0 and h, we obtain

$$\begin{aligned}\int_0^h u_i^*(x',z,t)dz \quad = \quad & hs_i^* + \frac{1}{2}h^2\tau_i^* + \frac{\hat{\alpha}}{\mu}\int_0^h\int_0^z\int_0^\zeta |u_i^*|^{p-2}u_i^*(x',\eta,t)d\eta d\zeta dz \\ & -\frac{1}{\mu}\int_0^h\int_0^z\int_0^\zeta \hat{f}_i(x',\eta,t)d\eta d\zeta dz.\end{aligned} \tag{47}$$

From (46) and (47), we deduce

$$\int_0^h u_i^*(x',z,t)dz - \frac{h}{2}s^* + F_i + U_i = 0,$$

with

$$\tilde{F}_i(x',h,t) = \frac{1}{\mu}\int_0^h F_i(x',z,t)dz - \frac{h}{2\mu}F_i(x',h,t), \; F_i(x',z,t) = \int_0^z\int_0^\zeta \hat{f}_i(x',\eta,t)d\eta d\zeta,$$

$$\tilde{U}_i(x',h,t) = -\frac{\hat{\alpha}}{\mu}\int_0^h U_i(x',z,t)dz + \frac{\hat{\alpha}h}{2\mu}U_i(x',h,t),\ U_i(x',z,t) = \int_0^z \int_0^\zeta |u_i^*|^{p-2}u_i^*(x',\eta,t)d\eta d\zeta.$$

Thus, we obtain the weak formulation

$$\int_\omega \left(\int_0^h u^*(x',z,t)dz - \frac{h}{2}s^* + \tilde{F} + \tilde{U}\right)\nabla\varphi(x')dx' = 0.$$

□

Theorem 5.4. *The solution (u_1^*, u_2^*) of the limit problem (39) and (40) is unique in $L^2(0,T,V_z) \cap L^2(0,T,L^p(\Omega)^2)$.*

Proof. Suppose there exist two solutions u^* and u^{**} of the variational formulation (39), thus we have

$$\mu\sum_{i=1}^2 \int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial \hat{\varphi}_i}{\partial z}dx'dz + \hat{\alpha}\sum_{i=1}^2\int_\Omega |u_i^*|^{p-2}u_i^*\hat{\varphi}_i dx'dz + \hat{J}_q(u^*,\hat{\varphi}) \tag{48}$$
$$= \sum_{i=1}^2\int_\Omega \hat{f}_i\hat{\varphi}_i dx'dz,$$

and

$$\mu\sum_{i=1}^2 \int_\Omega \frac{\partial u_i^{**}}{\partial z}\frac{\partial \hat{\varphi}_i}{\partial z}dx'dz + \hat{\alpha}\sum_{i=1}^2\int_\Omega |u_i^{**}|^{p-2}u_i^{**}\hat{\varphi}_i dx'dz + \hat{J}_q(u^*,\hat{\varphi}) \tag{49}$$
$$= \sum_{i=1}^2\int_\Omega \hat{f}_i\hat{\varphi}_i dx'dz.$$

Taking $\hat{\varphi} = u^{**} - u^*$ in (48), then $\hat{\varphi} = u^* - u^{**}$ in (49) and summing the two equations, we obtain

$$\mu\sum_{i=1}^2\int_\Omega \left|\frac{\partial\left(u_i^* - u_i^{**}\right)}{\partial z}\right|^2 dx'dz + \hat{\alpha}\sum_{i=1}^2\int_\Omega \left(|u_i^*|^{p-2}u_i^* - |u_i^{**}|^{p-2}u_i^{**}\right)(u_i^* - u_i^{**})dx'dz$$
$$+ \hat{J}_q(u^*, u^* - u^{**}) - \hat{J}_q(u^{**}, u^* - u^{**})$$
$$= 0.$$

On the other hand, we have

$$\left(|u_i^*|^{p-2}u_i^* - |u_i^{**}|^{p-2}u_i^{**}\right)(u_i^* - u_i^{**}) \geq 0,$$

and

$$\hat{J}_q(u^*, u^* - u^{**}) - \hat{J}_q(u^{**}, u^* - u^{**}) \geq 0.$$

By setting $\bar{W}(t) = u^*(t) - u^{**}(t)$, we get

$$\|\frac{\partial \bar{W}}{\partial z}\|^2_{L^2(\Omega)^2} = 0.$$

Using Poincaré's inequality, we deduce that

$$\|\bar{W}\|_{L^2(0,T,V_z)} = 0.$$

□

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Declarations

The author declares that he has no conflicts of interest.

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