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Weak Solutions to Leray-Lions Type Elliptic Equations with Variable Exponents

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Abstract

This work investigates a class of nonlinear elliptic problems posed on a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$*, described by the partial differential equation*

$$-\operatorname{div}\big(H(x, \nabla Z)\big) = F,$$

where H *is an operator of Leray–Lions type acting from the space* $W_0^{1,\alpha(\cdot)}(A)$ *into its dual.*
When the right-hand side F *belongs to* $L^{\theta(\cdot)}(A)$*, with* $\theta(\cdot) > 1$ *satisfying certain conditions, we prove the existence of weak solutions for this class of problems under* $\alpha(\cdot)$*-growth conditions. Our approach is based on a combination of variational methods, approximation techniques, and compactness arguments. The functional framework involves Sobolev spaces with variable exponents* $W_0^{1,\alpha(\cdot)}(A)$*, as well as Lebesgue spaces with variable exponents* $L^{\alpha(\cdot)}(A)$*. .*

Keywords

Nonlinear elliptic equations; Leray-Lions operator; Variable exponents; Weak solution; Variational methods; Irregular Data.

2020 Mathematics Subject Classification

Primary 35J60, 46E35 · Secondary 35D30, 35J20 ·

1. Introduction

This paper is devoted to the analysis of weak solutions for a class of nonlinear elliptic problems involving variable-exponent operators. A prototypical example is the boundary value problem:

$$\begin{cases} -\operatorname{div}\left(|\nabla Z|^{\alpha(\cdot)-2}\nabla Z\right) = F, & \text{in } \mathcal{A}, \\ Z = 0, & \text{on } \partial\mathcal{A}, \end{cases} \tag{1}$$

where $\mathcal{A} \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\mathcal{A}$, and the right-hand side F belongs to $L^{\theta(\cdot)}(\mathcal{A})$, with the exponent $\theta(\cdot)$ satisfying the constraints given in (7).

Equation (1) represents a generalization of the classical α-Laplace equation, where the constant exponent $\alpha \in (1, +\infty)$ is replaced by a variable exponent $\alpha(\cdot)$. This problem possesses a variational structure in the case when $\theta(\cdot) = \alpha'(\cdot)$

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where $\alpha'(\cdot)$ denotes the conjugate exponent of $\alpha(\cdot)$, defined through the relation $\alpha'(\cdot) = \frac{\alpha(\cdot)}{\alpha(\cdot)-1}$, wherein weak solutions correspond to critical points of the energy functional:

$$\mathcal{T}(Z) := \int_{\mathcal{A}} \frac{1}{\alpha(x)} |\nabla Z|^{\alpha(x)} dx - \int_{\mathcal{A}} F Z dx.$$

More generally, this work investigates an extended class of nonlinear elliptic problems with variable-exponent nonlinearities of the form:

$$\begin{cases} -\operatorname{div}\left(H(x, \nabla Z)\right) = F, & \text{in } \mathcal{A}, \\ Z = 0, & \text{on } \partial\mathcal{A}, \end{cases} \tag{2}$$

where $H : \mathcal{A} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Leray-Lions type operator. Specifically, H is a Carathéodory function satisfying, for almost every $x \in \mathcal{A}$ and for all $\eta, \eta' \in \mathbb{R}^N$, the following conditions:

$$H(x, \eta) \cdot \eta \geq c_1 |\eta|^{\alpha(x)}, \quad H(x, \eta) = (H_1, \ldots, H_N), \tag{3}$$

$$|H(x, \eta)| \leq c_2 \left(g(x) + |\eta|^{\alpha(x)-1} \right), \tag{4}$$

$$\big(H(x, \eta) - H(x, \eta')\big) \cdot (\eta - \eta') > 0, \quad \text{for } \eta \neq \eta', \tag{5}$$

where $c_1, c_2 > 0$ are constants, g is a non-negative function in $L^{\alpha'(\cdot)}(\mathcal{A})$.

The variable exponents $\theta(\cdot) : \overline{\mathcal{A}} \to (1, +\infty)$ and $\alpha(\cdot) : \overline{\mathcal{A}} \to (1, +\infty)$ are assumed to be continuous and satisfy the constraints:

$$1 + \frac{1}{\theta(x)} - \frac{1}{N} < \alpha(x) < N, \quad \text{for all } x \in \overline{\mathcal{A}}, \tag{6}$$

and

$$1 < \theta(x) < \frac{N\alpha(x)}{N\alpha(x) - N + \alpha(x)}, \quad \nabla\theta \in L^\infty(\mathcal{A}), \quad \text{for all } x \in \overline{\mathcal{A}}. \tag{7}$$

These conditions ensure the well-posedness of the Leray-Lions operator and its suitability for analysis via variational methods; we refer to [15] for further details on this approach.

The natural functional setting for such problems is provided by variable exponent Lebesgue-Sobolev spaces, denoted by $L^{\alpha(\cdot)}(\mathcal{A})$ and $W_0^{1,\alpha(\cdot)}(\mathcal{A})$. In recent years, the study of differential equations with variable exponents has emerged as a highly active research area, motivated by their diverse applications in fields such as electro-rheological fluids [13, 14, 16] and image processing [6]. We refer to [1, 7] and the references therein for an overview of the extensive literature on this subject.

The investigation of problem (2) represents a significant advancement, as the $\alpha(x)$-Laplacian generalizes its classical counterpart. In the constant exponent case where $2 - \frac{1}{N} < \alpha(\cdot) = \alpha$ and the right-hand side is a bounded Radon measure, problem (2) was studied by [5]. Building upon the works of [2, 3, 4, 11, 12], we establish the existence of weak solutions for problem (2) (Theorem 3.2) with right-hand sides in $L^{\theta(\cdot)}(\mathcal{A})$, where $\theta(\cdot)$ and $\alpha(\cdot)$ satisfy assumptions (6)-(7). Condition (6) ensures the necessary integrability to define weak solutions and apply Sobolev embeddings, while (7) plays a fundamental role in deriving the a priori estimates crucial to our analysis, particularly in Lemma 3.1.

The main contributions of this work are threefold. First, we prove an existence result (Theorem 3.2) for irregular data $F \in L^{\theta(\cdot)}(\mathcal{A})$ that may not belong to the dual space $\left(W_0^{1,\alpha(\cdot)}(\mathcal{A})\right)'$, extending previous studies that typically require F to be in this dual space. Second, the conditions imposed on $\theta(x)$ in (7) are less restrictive than those commonly found in the literature, thereby encompassing a broader class of right-hand side functions. Third, from a methodological perspective, we combine approximation techniques to handle irregular data with variational methods to exploit the problem's structure and establish uniqueness.

The proof of Theorem 3.2 relies on several key steps. We first establish crucial estimates in $W_0^{1,\beta(\cdot)}(\mathcal{A})$ for solutions of (2) (Lemma 3.1) using approximation techniques and compactness arguments. The core of the proof involves deriving uniform estimates for solutions of an approximate problem and subsequently passing to the limit. For the specific case of problem (1), we employ variational methods to establish existence, while the strict monotonicity condition (5) guarantees uniqueness.

To demonstrate the practical relevance of our theoretical framework, we conclude with an application to electro-rheological fluids. These smart materials exhibit reversible changes in their rheological properties under external electric fields, behavior that can be effectively modeled using variable-exponent PDEs. Consider the steady, laminar flow of an incompressible electro-rheological fluid between two parallel plates located at $y = \pm L$. Under a perpendicular electric field $\vec{E} = (0, E(y), 0)$, the apparent viscosity follows a power-law relation:

$$\mu(|\vec{E}|, |\nabla u|) = \mu_0(|\vec{E}|) |\nabla u|^{\alpha(|\vec{E}|)-2},$$

where $u(y)$ denotes the flow velocity and $\mu_0(|\vec{E}|) > 0$ is a field-dependent consistency coefficient. For non-homogeneous fields, the exponent becomes spatially dependent, $\alpha = \alpha(y)$. With a constant pressure gradient $F = -\frac{\partial p}{\partial x} > 0$ driving the flow, the momentum equation reduces to:

$$-\frac{d}{dy}\left(\mu_0(y)\left|\frac{du}{dy}\right|^{\alpha(y)-2}\frac{du}{dy}\right)=F,\quad \text{in } \mathcal{A}=(-L,L).$$

Defining $H(y,\eta)=\mu_0(y)|\eta|^{\alpha(y)-2}\eta$, this equation takes the form of our general problem (2). The operator H satisfies conditions (3)-(5), and the no-slip boundary condition at the plates yields the Dirichlet condition $u=0$ on $y=\pm L$. In this context, Theorem 3.2 guarantees the existence and uniqueness of a velocity field $u\in W_0^{1,\beta(\cdot)}(-L,L)$, even for irregular pressure gradients $F\in L^{\theta(\cdot)}$ not necessarily in the dual space, as may occur in complex pumping scenarios or near actuators.

Throughout the paper, the symbols C and C_i (for $i=1,2,\dots$) denote generic positive constants that may vary from line to line but remain independent of the approximation parameter n.

2. Variable exponent Lebesgue-Sobolev spaces

In this section, we recall fundamental concepts and results concerning variable exponent function spaces. Specifically, we examine the Lebesgue space $L^{\alpha(\cdot)}(\mathcal{A})$, the Sobolev space $W^{1,\alpha(\cdot)}(\mathcal{A})$, and its zero-trace subspace $W_0^{1,\alpha(\cdot)}(\mathcal{A})$, where $\mathcal{A}\subseteq\mathbb{R}^N$ is an open domain. For comprehensive treatments of these spaces and their theoretical foundations, we direct the reader to [7] and [9].

Let $p:\overline{\mathcal{A}}\to[1,\infty)$ be a continuous function. We denote by $L^{\alpha(\cdot)}(\mathcal{A})$ the space of measurable functions $Z(x)$ on $\mathcal{A}$ such that

$$\rho_{\alpha(\cdot)}(Z)=\int_{\mathcal{A}}|Z(x)|^{\alpha(x)}\,dx<+\infty.$$

The space $L^{\alpha(\cdot)}(\mathcal{A})$, equipped with the norm

$$\|Z\|_{\alpha(\cdot)}:=\|Z\|_{L^{\alpha(\cdot)}(\mathcal{A})}=\inf\left\{\varepsilon>0\mid\rho_{\alpha(\cdot)}(Z/\varepsilon)\leq 1\right\},$$

becomes a Banach space. Moreover, if $\alpha^-=\inf_{x\in\overline{\mathcal{A}}}\alpha(x)>1$, then $L^{\alpha(\cdot)}(\mathcal{A})$ is reflexive, and its dual space can be identified with $L^{\alpha'(\cdot)}(\mathcal{A})$, where $\frac{1}{\alpha(\cdot)}+\frac{1}{\alpha'(\cdot)}=1$. For any $Z\in L^{\alpha(\cdot)}(\mathcal{A})$ and $W\in L^{\alpha'(\cdot)}(\mathcal{A})$, the following Hölder-type inequality holds:

$$\left|\int_{\mathcal{A}}ZW\,dx\right|\leq\left(\frac{1}{\alpha^-}+\frac{1}{\alpha'^-}\right)\|Z\|_{\alpha(\cdot)}\|W\|_{\alpha'(\cdot)}\leq 2\|Z\|_{\alpha(\cdot)}\|W\|_{\alpha'(\cdot)}.$$

We define the Banach space $W_0^{1,\alpha(\cdot)}(\mathcal{A})$ as

$$W_0^{1,\alpha(\cdot)}(\mathcal{A})=\left\{Z\in L^{\alpha(\cdot)}(\mathcal{A})\mid|\nabla Z|\in L^{\alpha(\cdot)}(\mathcal{A})\text{ and }Z=0\text{ on }\partial\mathcal{A}\right\},$$

endowed with the norm $\|Z\|_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}=\|\nabla Z\|_{\alpha(\cdot)}$. The space $W_0^{1,\alpha(\cdot)}(\mathcal{A})$ is separable and reflexive provided that $1<\alpha^-\leq\alpha^+<\infty$. In general, smooth functions are not dense in $W_0^{1,\alpha(\cdot)}(\mathcal{A})$. However, if the variable exponent $\alpha(x)>1$ is logarithm Hölder continuous, i.e.,

$$|\alpha(v)-\alpha(w)|\leq-\frac{C}{\ln(|v-w|)}\quad\forall v,w\in\mathcal{A}\text{ such that }|v-w|\leq\frac{1}{2},$$

with $C>0$ then smooth functions are dense in $W_0^{1,\alpha(\cdot)}(\mathcal{A})$.

For $Z\in W_0^{1,\alpha(\cdot)}(\mathcal{A})$ with $\alpha\in C(\overline{\mathcal{A}},[1,+\infty))$, the Poincaré inequality holds:

$$\|Z\|_{\alpha(\cdot)}\leq C\|\nabla Z\|_{\alpha(\cdot)},\tag{8}$$

where C is a constant depending on $\mathcal{A}$ and the function α.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{\alpha(\cdot)}$ of the space $L^{\alpha(\cdot)}(\mathcal{A})$. We have the following result:

Lemma 2.1 ([9]). *If $(Z_n),Z\in L^{\alpha(\cdot)}(\mathcal{A})$, then the following relations hold:*

- $\|Z\|_{\alpha(\cdot)}<1\ (>1;=1)\Leftrightarrow\rho_{\alpha(\cdot)}(Z)<1\ (>1;=1)$,
- $\min\left(\rho_{\alpha(\cdot)}(Z)^{\frac{1}{\alpha^+}},\rho_{\alpha(\cdot)}(Z)^{\frac{1}{\alpha^-}}\right)\leq\|Z\|_{\alpha(\cdot)}\leq\max\left(\rho_{\alpha(\cdot)}(Z)^{\frac{1}{\alpha^+}},\rho_{\alpha(\cdot)}(Z)^{\frac{1}{\alpha^-}}\right)$,
- $\|Z\|_{\alpha(\cdot)}\leq\rho_{\alpha(\cdot)}(Z)+1$,
- $\|Z_n-Z\|_{\alpha(\cdot)}\to 0\Leftrightarrow\rho_{\alpha(\cdot)}(Z_n-Z)\to 0$, *provided* $\alpha^+<\infty$.

An important embedding result is as follows:

Lemma 2.2 ([8]). *Let $\mathcal{A}\subset\mathbb{R}^N$ be an open bounded set with Lipschitz boundary, and let $p:\overline{\mathcal{A}}\to(1,N)$ satisfy the logarithm Hölder continuity condition* (2). *Then, we have the following continuous embedding:*

$$W^{1,\alpha(\cdot)}(\mathcal{A})\hookrightarrow L^{\alpha^\star(\cdot)}(\mathcal{A}),$$

where $\alpha^\star(\cdot)=\frac{N\alpha(\cdot)}{N-\alpha(\cdot)}$.

3. Existence result

Definition 3.1. A function Z is a weak solution of problem (2) if

$$Z \in W_0^{1,1}(\mathcal{A}), \quad H(x,\nabla Z) \in (L^1(\mathcal{A}))^N,$$

and

$$\int_{\mathcal{A}} H(x,\nabla Z)\nabla\psi\,dx = \int_{\mathcal{A}} F\psi\;dx, \quad \forall \psi \in C_0^\infty(\mathcal{A}).$$

Our main result is the following:

Theorem 3.2. *Under the assumptions (3)-(6) and assume that (2) holds. If $F \in L^{\theta(\cdot)}(\mathcal{A})$ with $\theta(\cdot)$ as in (7) then the problem (2) has at least one weak solution $Z \in W_0^{1,\beta(\cdot)}(\mathcal{A})$, where $\beta(\cdot)$ is a continuous function on $\overline{\mathcal{A}}$ satisfying*

$$1 \le \beta(x) < \frac{N\theta(x)(\alpha(x)-1)}{N-\theta(x)} \quad \text{for all} \quad x \in \overline{\mathcal{A}}. \tag{9}$$

3.1. The approximation method for (2)

In this part, we employ the approximation method to study the existence of weak solutions for the problem (2).

Proof of Theorem 3.2. The proof needs three steps.
Step 1: Approximate problem
By the density property, we can choose a sequence $(F_n)_n \subset C_0^\infty(\mathcal{A})$

$$F_n \longrightarrow F \quad \text{strongly in } L^{\theta(\cdot)}(\mathcal{A}), \text{as } n \longrightarrow \infty.$$

such that

$$\|F_n\|_{L^{\theta(\cdot)}(\mathcal{A})} \le \|F\|_{L^{\theta(\cdot)}(\mathcal{A})}, \quad \forall n \ge 1. \tag{10}$$

For $Z \in W_0^{1,p(\cdot)}(\mathcal{A})$, we put

$$\mathcal{T}(Z) = -\text{div}\,(H(x,\nabla Z)).$$

The operator $\mathcal{T}$ maps $W_0^{1,\alpha(\cdot)}(\mathcal{A})$ into $\left(W_0^{1,\alpha(\cdot)}(\mathcal{A})\right)'$, thanks (5) $\mathcal{T}$ is monotone. The growth condition (4) implies that T is hemicontinuous.
i.e., for all Z, V, $W \in W_0^{1,\alpha(\cdot)}(\mathcal{A})$, the mapping $\mathbb{R} \ni \lambda \mapsto \langle \mathcal{T}(Z+\lambda V), W\rangle$ is continuous.
By (3) and Lemma 2.2 [7], we can write

$$\begin{aligned} \frac{\langle \mathcal{T}Z, Z\rangle}{\|Z\|_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}} &\ge c_1 \frac{\rho_{\alpha(\cdot)}(\nabla Z)}{\|Z\|_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}} \\ &\ge c_1 \frac{\min\left\{\|Z\|^{\alpha^+}_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}, \|Z\|^{\alpha^-}_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}\right\}}{\|Z\|_{W_0^{1,\alpha(\cdot)}(\mathcal{A})}}, \end{aligned}$$

this prove that $\mathcal{T}$ is coercive. By (4), we get the operator $\mathcal{T}$ is bounded.
Thus, we get the desired result.
Consequently, there exists at least one weak solution $(Z_n)_{n\in\mathbb{N}} \subset W_0^{1,\alpha(\cdot)}(\mathcal{A})$ (cf. J.L. Lions [10] , Theorem 2.7, page 180) satisfying

$$\int_{\mathcal{A}} H(x,\nabla Z_n)\nabla\psi\,dx = \int_{\mathcal{A}} F_n\psi\,dx, \quad \forall \psi \in W_0^{1,\alpha(\cdot)}(\mathcal{A}). \tag{11}$$

Step 2: Uniform estimates
We prove the following estimates: Let $\alpha(\cdot)$ be as in (6), and $\theta(\cdot)$ as in (7) with $\theta^- = \inf_{x\in\overline{\mathcal{A}}}\theta(x) > 1$. Then, for any constant $0<\sigma<1$, there exists a constant C_σ independent of n such that

$$\int_{\mathcal{A}} \frac{|\nabla Z_n|^{\alpha(x)}}{(1+|Z_n|)^\sigma}\,dx \le C_\sigma \left(1 + \left(\int_{\mathcal{A}} (1+|Z_n|)^{(1-\sigma)\frac{\theta^-}{\theta^- - 1}}\,dx\right)^{1-\frac{1}{\theta^-}}\right). \tag{12}$$

Proof of Lemma 3.1. For any given $0<\sigma<1$, define the function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ by

$$\phi_\sigma(s) = \int_0^s \frac{1}{(1+|t|)^\sigma}dt.$$

This function is well-defined, absolutely continuous, and satisfies $\phi_\sigma(0)=0$. Its derivative is

$$\phi_\sigma'(s) = \frac{1}{(1+|s|)^\sigma}, \quad \text{which implies} \quad |\phi_\sigma'(s)| \le 1 \quad \text{for all } s \in \mathbb{R}.$$

Since $Z_n \in W_0^{1,\alpha(\cdot)}(\mathcal{A})$ and ϕ_σ is Lipschitz, the composition $\phi_\sigma(Z_n)$ is an admissible test function in $W_0^{1,\alpha(\cdot)}(\mathcal{A})$. Substituting $\psi = \phi_\sigma(Z_n)$ into the weak formulation for the approximate problem (11) yields:

$$\int_{\mathcal{A}} H(x, \nabla Z_n) \cdot \nabla \phi_\sigma(Z_n) dx = \int_{\mathcal{A}} F_n \phi_\sigma(Z_n) dx. \tag{13}$$

Using the chain rule, we have $\nabla \phi_\sigma(Z_n) = \phi_\sigma'(Z_n) \nabla Z_n = \frac{\nabla Z_n}{(1+|Z_n|)^\sigma}$. Substituting this into the left-hand side of (13) and applying the coercivity condition (3) gives:

$$\begin{aligned} \int_{\mathcal{A}} H(x, \nabla Z_n) \cdot \nabla \phi_\sigma(Z_n) dx &= \int_{\mathcal{A}} H(x, \nabla Z_n) \cdot \frac{\nabla Z_n}{(1+|Z_n|)^\sigma} dx \\ &\geq c_1 \int_{\mathcal{A}} \frac{|\nabla Z_n|^{\alpha(x)}}{(1+|Z_n|)^\sigma} dx. \end{aligned}$$

Thus, we obtain the lower bound:

$$\int_{\mathcal{A}} \frac{|\nabla Z_n|^{\alpha(x)}}{(1+|Z_n|)^\sigma} dx \leq \frac{1}{c_1} \int_{\mathcal{A}} F_n \phi_\sigma(Z_n) dx. \tag{14}$$

We now estimate the integral $I := \int_{\mathcal{A}} F_n \phi_\sigma(Z_n) dx$. First, we derive a pointwise bound for $\phi_\sigma(s)$. For $s \geq 0$, we have:

$$\phi_\sigma(s) = \int_0^s \frac{1}{(1+t)^\sigma} dt = \frac{1}{1-\sigma} \left((1+s)^{1-\sigma} - 1 \right) \leq \frac{1}{1-\sigma} (1+s)^{1-\sigma}.$$

By symmetry, for all $s \in \mathbb{R}$, it holds that:

$$|\phi_\sigma(s)| \leq \frac{1}{1-\sigma} (1+|s|)^{1-\sigma}.$$

Using this bound and Hölder's inequality for variable exponents with the pair $\theta(\cdot)$ and its conjugate $\theta'(\cdot) = \frac{\theta(\cdot)}{\theta(\cdot)-1}$, we estimate:

$$\begin{aligned} I &= \int_{\mathcal{A}} F_n \phi_\sigma(Z_n) dx \\ &\leq \frac{1}{1-\sigma} \int_{\mathcal{A}} |F_n| (1+|Z_n|)^{1-\sigma} dx \\ &\leq \frac{1}{1-\sigma} \|F_n\|_{L^{\theta(\cdot)}(\mathcal{A})} \|(1+|Z_n|)^{1-\sigma}\|_{L^{\theta'(\cdot)}(\mathcal{A})}. \end{aligned}$$

From the uniform bound, we have $\|F_n\|_{L^{\theta(\cdot)}(\mathcal{A})} \leq \|F\|_{L^{\theta(\cdot)}(\mathcal{A})}$. Let $M_F := \|F\|_{L^{\theta(\cdot)}(\mathcal{A})}$, which is a constant independent of n. Therefore,

$$I \leq \frac{M_F}{1-\sigma} \|(1+|Z_n|)^{1-\sigma}\|_{L^{\theta'(\cdot)}(\mathcal{A})}. \tag{15}$$

To proceed, we relate the Luxemburg norm to its corresponding modular. Recall that for a function $f \in L^{p(\cdot)}(\mathcal{A})$, we have the general property:

$$\|f\|_{L^{p(\cdot)}(\mathcal{A})} \leq \rho_{p(\cdot)}(f) + 1,$$

where $\rho_{p(\cdot)}(f) = \int_{\mathcal{A}} |f(x)|^{p(x)} dx$. Applying this with $f = (1+|Z_n|)^{1-\sigma}$ and $p(\cdot) = \theta'(\cdot)$, we get:

$$\|(1+|Z_n|)^{1-\sigma}\|_{L^{\theta'(\cdot)}(\mathcal{A})} \leq \int_{\mathcal{A}} (1+|Z_n|)^{(1-\sigma)\theta'(x)} dx + 1.$$

A key observation is that since $\theta^- > 1$, we have $\theta'(x) = \frac{\theta(x)}{\theta(x)-1} \leq \frac{\theta^+}{\theta^- - 1}$. However, a more precise uniform bound is needed. Notice that the function $t \mapsto t/(t-1)$ is decreasing for $t > 1$. Therefore, for all $x \in \mathcal{A}$,

$$\theta'(x) \leq \frac{\theta^-}{\theta^- - 1}.$$

Let $r := \frac{\theta^-}{\theta^- - 1} > 1$. Since $(1-\sigma) > 0$, the function $s \mapsto s^{(1-\sigma)}$ is increasing for $s \geq 0$. Thus, for all $x \in \mathcal{A}$,

$$(1+|Z_n(x)|)^{(1-\sigma)\theta'(x)} \leq (1+|Z_n(x)|)^{(1-\sigma)r} + 1.$$

The addition of 1 accounts for the possibility that the exponent $(1-\sigma)\theta'(x)$ could be less than 1, ensuring the inequality holds pointwise. Therefore,

$$\int_{\mathcal{A}} (1+|Z_n|)^{(1-\sigma)\theta'(x)} dx \leq \int_{\mathcal{A}} \left((1+|Z_n|)^{(1-\sigma)r} + 1 \right) dx = \int_{\mathcal{A}} (1+|Z_n|)^{(1-\sigma)r} dx + |\mathcal{A}|.$$

Combining this with (15) and the previous norm estimate yields:

$$I \leq \frac{M_F}{1-\sigma} \left(\int_{\mathcal{A}} (1+|Z_n|)^{(1-\sigma)r} dx + |\mathcal{A}| + 1 \right). \tag{16}$$

We now apply the elementary inequality: for $W_1, W_2 \geq 0$ and $r > 0$,

$$(W_1 + W_2)^r \leq C_r(W_1^r + W_2^r), \quad \text{where } C_r = \max\{1, 2^{r-1}\}.$$

Let $W_1 = 1$, $W_2 = |Z_n|$, and apply the inequality with exponent $(1-\sigma)r$:

$$(1 + |Z_n|)^{(1-\sigma)r} \leq C_{(1-\sigma)r}\left(1 + |Z_n|^{(1-\sigma)r}\right).$$

Substituting this into (16) gives:

$$I \leq \frac{M_F}{1-\sigma} C_{(1-\sigma)r}\left(\int_{\mathcal{A}}\left(1 + |Z_n|^{(1-\sigma)r}\right)dx + |\mathcal{A}| + 1\right) \leq C'\left(1 + \int_{\mathcal{A}} |Z_n|^{(1-\sigma)r} dx\right),$$

where $C' = \frac{M_F}{1-\sigma} C_{(1-\sigma)r}(2|\mathcal{A}| + 1)$ is a constant independent of n.

Substituting the estimate for I back into the inequality (14), we obtain:

$$\int_{\mathcal{A}} \frac{|\nabla Z_n|^{\alpha(x)}}{(1 + |Z_n|)^{\sigma}} dx \leq \frac{C'}{c_1}\left(1 + \int_{\mathcal{A}} |Z_n|^{(1-\sigma)r} dx\right).$$

Recalling that $r = \frac{\theta^-}{\theta^- - 1}$, this is precisely the desired estimate (12) with the constant $C_\sigma = \frac{C'}{c_1}$, which is independent of n. $\square$

Let $\alpha(\cdot)$ as in (6), and $\theta(\cdot)$ as in (7), and $F \in L^{\theta(\cdot)}(\mathcal{A})$. Then there exists a constant C_1 such that

$$\|Z_n\|_{W_0^{1,\beta(\cdot)}(\mathcal{A})} \leq C_1,$$

for all continuous functions $\beta(\cdot)$ as in (9).

Remark 3.3. Note that the result given in Lemma 3.1 also holds for any measurable function $r : \overline{\mathcal{A}} \to \mathbb{R}$ such that

$$ess \inf_{x \in \overline{\mathcal{A}}} \left(\frac{N\theta(x)(\alpha(x) - 1)}{N - \theta(x)} - r(x)\right) > 0.$$

Indeed, in both cases there exists a continuous function $\varpi : \overline{\mathcal{A}} \to \mathbb{R}$ such that for almost every $x \in \overline{\mathcal{A}}$:

$$r(x) \leq \varpi(x) \leq \frac{N\theta(x)(\alpha(x) - 1)}{N - \theta(x)}.$$

From Lemma 3.1, we deduce, in both cases, that $(Z_n)_n$ is bounded in $W_0^{1,\varpi(\cdot)}(\mathcal{A})$. Finally, by the continuous embedding $W_0^{1,\varpi(\cdot)}(\mathcal{A}) \hookrightarrow W_0^{1,\beta(\cdot)}(\mathcal{A})$, we have the desired result.

Proof of Lemma 3.1. Firstly, note that since $\theta(\cdot) > 1$ and $\alpha(\cdot)$ is defined as in (6), we get

$$1 < \frac{N\theta(x)(\alpha(x) - 1)}{N - \theta(x)}, \quad \text{for all} \quad x \in \overline{\mathcal{A}}.$$

Now, consider the following cases:

Case (a): Let β^+ be a constant satisfying

$$\beta^+ < \frac{N\theta^-(\alpha^- - 1)}{N - \theta^-}. \tag{17}$$

Note that the assumption (7) implies that

$$\frac{N\theta^-(\alpha^- - 1)}{N - \theta^-} < \alpha^-. \tag{18}$$

Using Hölder's inequality with (12), we obtain

$$\begin{aligned} \int_{\mathcal{A}} |\nabla Z_n|^{\beta^+} dx &= \int_{\mathcal{A}} \frac{|\nabla Z_n|^{\beta^+}}{(1 + |Z_n|)^{\sigma\frac{\beta^+}{\alpha^-}}}(1 + |Z_n|)^{\sigma\frac{\beta^+}{\alpha^-}} dx \\ &\leq C_2\left(1 + \left(\int_{\mathcal{A}} (1 + |Z_n|)^{(1-\sigma)\frac{\theta^-}{\theta^- - 1}} dx\right)\right)^{(1-\frac{1}{\theta^-})\frac{\beta^+}{\alpha^-}} \cdot \left(1 + \left(\int_{\mathcal{A}} (1 + |Z_n|)^{\sigma\frac{\beta^+}{\alpha^- - \beta^+}} dx\right)\right)^{1-\frac{\beta^+}{\alpha^-}}, \end{aligned} \tag{19}$$

By (17) and (18) , we get

$$1 - \left(\frac{N\beta^+}{N - \beta^+}\right)\left(\frac{\theta^- - 1}{\theta^-}\right) < \frac{\theta^-(\alpha^- - \beta^+)}{(\theta^- - 1)\beta^+ + \theta^-(\alpha^- - \beta^+)} < 1. \tag{20}$$

Now, choose $\sigma \in (0,1)$ such that

$$\frac{\sigma\beta^+}{\alpha^- - \beta^+} < \frac{\theta^-(1-\sigma)}{\theta^- - 1} < \beta^{+\star} = \frac{N\beta^+}{N - \beta^+}. \tag{21}$$

Notice that (20) and (21) are respectively equivalent to

$$1-\left(\frac{N\beta^+}{N-\beta^+}\right)\left(\frac{\theta^- -1}{\theta^-}\right)<\sigma<\frac{\theta^-(\alpha^- -\beta^+)}{(\theta^- -1)\beta^+ +\theta^-(\alpha^- -\beta^+)}<1. \tag{22}$$

Therefore, by (19), (21) and using Sobolev inequality with $\beta^{+\star}$, we obtain

$$\begin{aligned}\int_{\mathcal{A}}|\nabla Z_n|^{\beta^+}\,dx &\le C_3\left(1+\int_{\mathcal{A}}|Z_n|^{\frac{\theta(1-\delta)}{\theta^- -1}}\,dx\right)^{1-\frac{\beta^+}{\theta^-\alpha^-}}\\ &\le C_4\left(1+\int_{\mathcal{A}}|Z_n|^{\beta^{+\star}}\,dx\right)^{1-\frac{\beta^+}{\theta^-\alpha^-}}\\ &\le C_5\left(1+\int_{\mathcal{A}}|\nabla Z_n|^{\beta^+}\,dx\right)^{(\frac{N}{N-\beta^+})(1-\frac{\beta^+}{\theta^-\alpha^-})}\\ &\le C_6+C_6\left(\int_{\mathcal{A}}|\nabla Z_n|^{\beta^+}\,dx\right)^{(\frac{N}{N-\beta^+})(1-\frac{\beta^+}{\theta^-\alpha^-})},\end{aligned} \tag{23}$$

By the fact that

$$\theta^-<\frac{N\alpha^-}{N\alpha^- -N+\alpha^-}<\frac{N}{\alpha^-}, \tag{24}$$

together with the assumption (17), this implies that

$$\beta^+<\theta^-\alpha^- \text{ and } 0<\left(\frac{N}{N-\beta^+}\right)\left(1-\frac{\beta^+}{\theta^-\alpha^-}\right)<1.$$

Hence, the estimate (23) imply that (∇Z_n) is bounded in $L^{\beta^+}(\mathcal{A})$.
Since $|\nabla Z_n|^{\beta(\cdot)}\le|\nabla Z_n|^{\beta^+}+1$, we obtain that (Z_n) is bounded in $W_0^{1,\beta(\cdot)}(\mathcal{A})$. This completes the proof in Case (a).

Case (b): Let β be a continuous function satisfying (9) and

$$\beta^+\ge\frac{N\theta^-(\alpha^- -1)}{N-\theta^-}.$$

By the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$ on $\overline{\mathcal{A}}$, there exists a constant $\eta>0$ such that

$$\max_{y\in O(x,\eta)\cap\mathcal{A}}\beta(y)<\min_{y\in O(x,\eta)\cap\mathcal{A}}\frac{N\theta^-(\alpha(y)-1)}{N-\theta^-}\quad\text{for all } x\in\overline{\mathcal{A}}. \tag{25}$$

Note that $\overline{\mathcal{A}}$ is compact and therefore we can cover it with a finite number of balls $(O_i)_{i=1,\dots,k}$. Moreover, there exists a constant $\rho>0$ such that

$$|\mathcal{A}_i|=\text{meas}(\mathcal{A}_i)>\rho,\ \mathcal{A}_i:=O_i\cap\mathcal{A},\quad\text{for all } i=1,\dots,k. \tag{26}$$

We denote by β_i^+ the local maximum of β on $\overline{\mathcal{A}_i}$ (respectively α_i^- the local minimum of α on $\overline{\mathcal{A}_i}$), such that

$$\beta_i^+<\frac{N\theta^-(\alpha_i^- -1)}{N-\theta^-}\quad\text{for all } i=1,\dots,k. \tag{27}$$

Using the same arguments as before locally, we obtain the similar estimate as in (23)

$$\int_{\mathcal{A}_i}|\nabla Z_n|^{\beta_i^+}\,dx\le C_7\left(1+\int_{\mathcal{A}_i}|Z_n|^{\beta_i^{+\star}}\,dx\right)^{1-\frac{\beta_i^+}{\theta^-\alpha_i^-}},\ \text{for all } i=1,\dots,k. \tag{28}$$

On the other hand, the Poincaré-Wirtinger inequality gives

$$\|Z_n-\widetilde{Z_n}\|_{L^{\beta_i^{+\star}}(\mathcal{A}_i)}\le C_8\|\nabla Z_n\|_{L^{\beta_i^+}(\mathcal{A}_i)}, \tag{29}$$

$$\text{where}\qquad \widetilde{Z_n}=\frac{1}{|\mathcal{A}_i|}\int_{\mathcal{A}_i}Z_n(x)\,dx,\quad \beta_i^{+\star}=\frac{N\beta_i^+}{N-\beta_i^+}.$$

Moreover, note that the sequence $(Z_n)_n$ is bounded in $L^1(\mathcal{A})$. So, from (26), we have

$$\|\widetilde{Z_n}\|_{L^1(\mathcal{A})}\le C_8,$$

Therefore, by (29), we deduce that

$$\begin{aligned}\|Z_n\|_{L^{\beta_i^{+\star}}(\mathcal{A}_i)}&\le\|Z_n-\widetilde{Z_n}\|_{L^{\beta_i^{+\star}}(\mathcal{A}_i)}+\|\widetilde{Z_n}\|_{L^{\beta_i^+}(\mathcal{A}_i)}\\ &\le C_8\|\nabla Z_n\|_{L^{\beta_i^+}(\mathcal{A}_i)}+C_9,\quad\text{for all}\quad i=1,\cdots,k.\end{aligned}$$

Thus, using (28), we obtain

$$\int_{\mathcal{A}_i} |\nabla Z_n|^{\beta_i^+}\, dx \le C_{10} + C_{10}\left(\int_{\mathcal{A}_i} |\nabla Z_n|^{\beta_i^+}\, dx\right)^{\left(\frac{N}{N-\beta_i^+}\right)\left(1-\frac{\beta_i^+}{\theta^-\alpha_i^-}\right)},$$

By (27) and arguing locally as in (24), we deduce

$$0 < \left(\frac{N}{N-\beta_i^+}\right)\left(1-\frac{\beta_i^+}{\theta^-\alpha_i^-}\right) < 1,$$

so that

$$\int_{\mathcal{A}_i} |\nabla Z_n|^{\beta_i^+}\, dx \le C_{11}, \text{ for all } i = 1, \dots, k.$$

Recall that

$$\beta(x) \le \beta_i^+, \quad \text{for all } x \in \mathcal{A}_i \text{ and for all } i = 1, \dots, k.$$

So, we get

$$\int_{\mathcal{A}_i} |\nabla Z_n|^{\beta(x)}\, dx \le \int_{\mathcal{A}_i} |\nabla Z_n|^{\beta_i^+}\, dx + |\mathcal{A}_i| \le C_{12}.$$

Since $\mathcal{A} \subset \bigcup_{i=1}^{N} \mathcal{A}_i$, for all $i = 1, \dots, k$. We deduce that

$$\int_{\mathcal{A}} |\nabla Z_n|^{\beta(x)}\, dx \le \sum_{i=1}^{k} \int_{\mathcal{A}_i} |\nabla Z_n|^{\beta(x)}\, dx \le C_{13}.$$

This finishes the proof of the Case(b). □

Remark 3.4. Remark that in the constant case and $F \in L^{\theta(\cdot)}(\mathcal{A})$, we choose in (19)

$$\sigma = \frac{\alpha N - \theta^-\alpha - \theta^- N\alpha + \theta^- N}{N - \theta^-\alpha} \in (0,1),$$

to obtain

$$\beta = \frac{\theta^- N(\alpha-1)}{N-\theta^-} \Longrightarrow (1-\sigma)\frac{\sigma^-}{\theta^- - 1} = \frac{\sigma\beta}{\alpha-\beta} = \frac{N\beta}{N-\beta},$$

It is easy to check that, instead of the global estimate (23), we find

$$\int_{\mathcal{A}} |\nabla Z_n|^{\beta}\, dx \le C + C\left(\int_{\mathcal{A}} |\nabla Z_n|^{\beta}\, dx\right)^{\left(\frac{N}{N-\beta}\right)\left(1-\frac{\beta}{\theta^-\alpha}\right)},$$

where $0 < \left(\frac{N}{N-\beta}\right)\left(1-\frac{\beta}{\theta^-\alpha}\right) < 1$. Then (2) has at least one weak solution Z, possesses the regularity $Z \in W_0^{1,\beta}(\mathcal{A})$ far all $\beta = \frac{N\theta^-(\alpha-1)}{N-\theta^-}$.

Step 3: Passage to the limit

From Lemma 3.1 together with the continuous embedding $W_0^{1,\beta(\cdot)}(\mathcal{A}) \hookrightarrow W_0^{1,\beta^-}(\mathcal{A})$, we have a subsequence (still denoted $(Z_n)_n$) such that

$$\begin{aligned} Z_n &\rightharpoonup Z \quad \text{weakly in} \quad W_0^{1,\beta^-}(\mathcal{A}),\\ Z_n &\to Z \quad \text{strongly in} \quad L^{\beta^-}(\mathcal{A}),\\ Z_n &\to Z \quad \text{a.e in} \quad \mathcal{A}. \end{aligned} \tag{30}$$

To complete the proof, we need the following lemmas: We have

$$\nabla Z_n \to \nabla Z \quad \text{a.e in} \quad \mathcal{A}. \tag{31}$$

Proof of Lemma 3.1. To establish the almost everywhere convergence of the gradients, we employ a strategy based on convergence in measure and Vitali's theorem. We first show that ∇Z_n converges to ∇Z in measure. That is, for every $\epsilon > 0$,

$$\lim_{n\to\infty} |\{x \in \mathcal{A} : |\nabla Z_n(x) - \nabla Z(x)| > \epsilon\}| = 0.$$

Consider the quantity:

$$I_n = \int_{\mathcal{A}} (H(x,\nabla Z_n) - H(x,\nabla Z))\cdot(\nabla Z_n - \nabla Z)dx.$$

Using the weak formulation (11) with test function $\psi = Z_n - Z$, we have:

$$\int_{\mathcal{A}} H(x, \nabla Z_n) \cdot (\nabla Z_n - \nabla Z) dx = \int_{\mathcal{A}} F_n (Z_n - Z) dx.$$

Similarly, by density arguments, we can use $Z_n - Z$ as a test function in the limit equation satisfied by Z to obtain:

$$\int_{\mathcal{A}} H(x, \nabla Z) \cdot (\nabla Z_n - \nabla Z) dx = \int_{\mathcal{A}} F (Z_n - Z) dx.$$

Subtracting these two equations gives:

$$I_n = \int_{\mathcal{A}} (F_n - F)(Z_n - Z) dx. \tag{32}$$

We now estimate the right-hand side of (32). By Hölder's inequality:

$$\left| \int_{\mathcal{A}} (F_n - F)(Z_n - Z) dx \right| \leq \|F_n - F\|_{L^{\theta(\cdot)}(\mathcal{A})} \|Z_n - Z\|_{L^{\theta'(\cdot)}(\mathcal{A})},$$

where $\theta'(\cdot) = \frac{\theta(\cdot)}{\theta(\cdot)-1}$ is the conjugate exponent.

Since $F_n \to F$ strongly in $L^{\theta(\cdot)}(\mathcal{A})$ and $Z_n \to Z$ strongly in $L^{\beta^-}(\mathcal{A})$, and noting that the continuous embedding $L^{\beta^-}(\mathcal{A}) \hookrightarrow L^{\theta'(\cdot)}(\mathcal{A})$ holds due to conditions (6)-(7), we conclude that:

$$\lim_{n \to \infty} I_n = 0.$$

Now we employ the strict monotonicity condition (5). For any $\epsilon > 0$, consider the set:

$$E_{n,\epsilon} = \{ x \in \mathcal{A} : |\nabla Z_n(x) - \nabla Z(x)| > \epsilon \}.$$

By the monotonicity condition (5), there exists $\delta(\epsilon) > 0$ such that for all $x \in E_{n,\epsilon}$:

$$(H(x, \nabla Z_n) - H(x, \nabla Z)) \cdot (\nabla Z_n - \nabla Z) \geq \delta(\epsilon) > 0.$$

Therefore, we have:

$$I_n \geq \int_{E_{n,\epsilon}} (H(x, \nabla Z_n) - H(x, \nabla Z)) \cdot (\nabla Z_n - \nabla Z) dx \geq \delta(\epsilon) |E_{n,\epsilon}|.$$

Since $I_n \to 0$ as $n \to \infty$, it follows that:

$$\lim_{n \to \infty} |E_{n,\epsilon}| = 0 \quad \text{for every } \epsilon > 0.$$

This establishes that $\nabla Z_n \to \nabla Z$ in measure.

To obtain almost everywhere convergence, we use the fact that convergence in measure implies the existence of a subsequence that converges almost everywhere. However, we need to show that the entire sequence converges almost everywhere.

Since ∇Z_n is bounded in $L^{\beta(\cdot)}(\mathcal{A})$ and $\beta^- > 1$, the sequence is uniformly integrable. By Vitali's theorem, convergence in measure combined with uniform integrability implies strong convergence in $L^1(\mathcal{A})$ of a subsequence. But we need a more refined argument for the entire sequence.

Consider the quantity:

$$J_n = \int_{\mathcal{A}} |(H(x, \nabla Z_n) - H(x, \nabla Z)) \cdot (\nabla Z_n - \nabla Z)| \, dx.$$

From the growth condition (4) and the boundedness of ∇Z_n in $L^{\beta(\cdot)}(\mathcal{A})$, we can show that J_n is uniformly bounded. Moreover, since $I_n \to 0$ and the integrand is non-negative, we actually have:

$$(H(x, \nabla Z_n) - H(x, \nabla Z)) \cdot (\nabla Z_n - \nabla Z) \to 0 \quad \text{in } L^1(\mathcal{A}).$$

Therefore, there exists a subsequence (still denoted by n) such that:

$$(H(x, \nabla Z_n) - H(x, \nabla Z)) \cdot (\nabla Z_n - \nabla Z) \to 0 \quad \text{a.e. in } \mathcal{A}.$$

Now, by the strict monotonicity condition (5), for almost every $x \in \mathcal{A}$, the mapping $\eta \mapsto H(x, \eta)$ is strictly monotone. This implies that if:

$$(H(x, \eta_n) - H(x, \eta)) \cdot (\eta_n - \eta) \to 0,$$

then necessarily $\eta_n \to \eta$.

Applying this pointwise with $\eta_n = \nabla Z_n(x)$ and $\eta = \nabla Z(x)$, we conclude that:

$$\nabla Z_n(x) \to \nabla Z(x) \quad \text{for almost every } x \in \mathcal{A}.$$

To show that the entire sequence converges almost everywhere (not just a subsequence), we use a standard argument. Suppose, for contradiction, that there exists a subset $E \subset \mathcal{A}$ with $|E| > 0$ such that for some $\epsilon > 0$ and infinitely many n:

$$|\nabla Z_n(x) - \nabla Z(x)| > \epsilon \quad \text{for all } x \in E.$$

Then by the same monotonicity argument, we would have:

$$I_n \geq \delta(\epsilon)|E| > 0 \quad \text{for infinitely many } n,$$

contradicting the fact that $I_n \to 0$. Therefore, the entire sequence ∇Z_n converges to ∇Z almost everywhere in $\mathcal{A}$.

This completes the proof of Lemma 3.1. □

We have

$$H(x, \nabla Z_n) \to H(x, \nabla Z) \quad \text{strongly in} \quad L^{\beta(\cdot)}(\mathcal{A}), \tag{33}$$

for some continuous function $\beta(\cdot) : \overline{\mathcal{A}} \to [1, \frac{N\theta(\cdot)}{N-\theta(\cdot)})$, where $\theta(\cdot)$ is a defined in (7).

Proof. To prove (33), we apply Vitali's theorem with taking in consideration Lemma 3.1, (30), (31), (4) and (6). □

Finally, for $\psi \in C_0^\infty(\mathcal{A})$, we have

$$\int_{\mathcal{A}} H(x, \nabla Z_n)\nabla\psi \, dx = \int_{\mathcal{A}} F_n \psi \, dx. \tag{34}$$

Using (33), we can pass to the limit for $n \to +\infty$ in the weak formulation (34), we obtain that Z is a weak solution for (2). □

Remark 3.5. Under the assumption $F \in L^{\theta(\cdot)}(\mathcal{A})$ in Theorem 3.2, we can deduce that F is never in the dual space $\left(W_0^{1,\alpha(\cdot)}(\mathcal{A})\right)'$, so that the result of this paper deals with irregular data. If $\theta(\cdot)$ tends to be 1, then $\beta(\cdot) = \frac{N\theta(\cdot)(\alpha(\cdot)-1)}{N-\theta(\cdot)}$ tends to be $\frac{N(\alpha(\cdot)-1)}{N-1}$.

3.2. The variational method for (1) with $\theta(\cdot) = \alpha'(\cdot)$

To prove that the problem (1) has a variational structure, meaning that weak solutions can be obtained as critical points of an energy functional, we need to follow these steps:

First we see that the energy functional $\mathcal{T} : W_0^{1,\alpha(\cdot)}(\mathcal{A}) \to \mathbb{R}$ associated with the problem (1) is typically defined as:

$$\mathcal{T}(Z) = \int_{\mathcal{A}} \frac{1}{\alpha(x)} |\nabla Z|^{\alpha(x)} \, dx - \int_{\mathcal{A}} FZ \, dx.$$

Here, the first term $\displaystyle\int_{\mathcal{A}} \frac{1}{\alpha(x)} |\nabla Z|^{\alpha(x)} \, dx$ represents the energy associated with the gradient of Z and the second term $\displaystyle\int_{\mathcal{A}} FZ \, dx$ represents the work done by the external force F.

In addition, to show that weak solutions correspond to critical points of $\mathcal{T}$, we compute the Gâteaux derivative of $\mathcal{T}$ in the direction of a test function $\phi \in W_0^{1,\alpha(\cdot)}(\mathcal{A})$. The Gâteaux derivative is given by:

$$\langle \mathcal{T}'(Z), \phi \rangle = \lim_{t \to 0} \frac{\mathcal{T}(Z + t\phi) - \mathcal{T}(Z)}{t}.$$

For the functional $\mathcal{T}$, this derivative can be computed explicitly as:

$$\langle \mathcal{T}'(Z), \phi \rangle = \int_{\mathcal{A}} |\nabla Z|^{\alpha(x)-2} \nabla Z \cdot \nabla\phi \, dx - \int_{\mathcal{A}} F\phi \, dx.$$

Second, we need to check that a function $Z \in W_0^{1,\alpha(\cdot)}(\mathcal{A})$ is a critical point of $\mathcal{T}$ if $\langle \mathcal{T}'(Z), \phi \rangle = 0$ for all $\phi \in W_0^{1,\alpha(\cdot)}(\mathcal{A})$. This condition is equivalent to the weak formulation of the problem:

$$\int_{\mathcal{A}} |\nabla Z|^{\alpha(x)-2} \nabla Z \cdot \nabla\phi \, dx = \int_{\mathcal{A}} F\phi \, dx \quad \text{for all } \phi \in W_0^{1,\alpha(\cdot)}(\mathcal{A}).$$

This is precisely the weak form of the equation:

$$-\text{div}\left(|\nabla Z|^{\alpha(x)-2} \nabla Z\right) = F \quad \text{in } \mathcal{A},$$

with the Dirichlet boundary condition $Z = 0$ on $\partial\mathcal{A}$.

To confirm that the problem (1) has a variational structure, we need to ensure that:

1. The energy functional $\mathcal{T}$ is well-defined and differentiable on $W_0^{1,\alpha(\cdot)}(\mathcal{A})$.
2. The critical points of $\mathcal{T}$ correspond to weak solutions of the problem.

These properties follow from the conditions (3)-(4) of the $\alpha(x)$-Laplacian operator beside this, the continuity and differentiability of the functional $\mathcal{T}$ in the variable exponent Sobolev space setting.

Moreover, to prove the existence of critical points (and hence weak solutions) of the problem (1), we can use (3)-(5) to show that $\mathcal{T}$ is bounded below and coercive, and then apply the direct method to find a minimizer.

In the end we need to prove the uniqueness of weak solutions of the problem (1) as follows:

Assume there exist two weak solutions $Z_1, Z_2 \in W_0^{1,\beta(\cdot)}(\mathcal{A})$ to the problem. Then, for all $\varphi \in W_0^{1,\beta(\cdot)}(\mathcal{A})$, we have:

$$\int_{\mathcal{A}} H(x, \nabla Z_1)\nabla\varphi\, dx = \int_{\mathcal{A}} F\varphi\, dx,$$

$$\int_{\mathcal{A}} H(x, \nabla Z_2)\nabla\varphi\, dx = \int_{\mathcal{A}} F\varphi\, dx.$$

Subtracting the two equations, we obtain:

$$\int_{\mathcal{A}} (H(x, \nabla Z_1) - H(x, \nabla Z_2))\, \nabla\varphi\, dx = 0, \quad \forall\varphi \in W_0^{1,\beta(\cdot)}(\mathcal{A}).$$

Let $\varphi = Z_1 - Z_2$. Since $Z_1, Z_2 \in W_0^{1,\beta(\cdot)}(\mathcal{A})$, it follows that $\varphi \in W_0^{1,\beta(\cdot)}(\mathcal{A})$. Substituting φ into the equation, we get:

$$\int_{\mathcal{A}} (H(x, \nabla Z_1) - H(x, \nabla Z_2))\, \nabla(Z_1 - Z_2)\, dx = 0.$$

By (5), we have:

$$(H(x, \nabla Z_1) - H(x, \nabla Z_2)) \cdot (\nabla Z_1 - \nabla Z_2) \geq 0,$$

with equality if and only if $\nabla Z_1 = \nabla Z_2$ almost everywhere in $\mathcal{A}$.

From the integral equation:

$$\int_{\mathcal{A}} (H(x, \nabla Z_1) - H(x, \nabla Z_2)) \cdot (\nabla Z_1 - \nabla Z_2)\, dx = 0,$$

and (5), it follows that:

$$\nabla Z_1 = \nabla Z_2 \quad \text{almost everywhere in } \mathcal{A}.$$

Since Z_1 and Z_2 have the same gradient $\nabla Z_1 = \nabla Z_2$ and both satisfy the Dirichlet boundary condition $Z_1 = Z_2 = 0$ on $\partial\mathcal{A}$, we conclude:

$$Z_1 = Z_2 \quad \text{almost everywhere in } \mathcal{A}.$$

Thus, the weak solution $Z \in W_0^{1,\beta(\cdot)}(\mathcal{A})$ of the problem (1) is unique.

Conclusion

In this work, we combined the approximation methods and the variational methods to study the existence of weak solutions for nonlinear elliptic equations with variable exponents. The approximation methods handled non-smooth data by constructing a sequence of approximate solutions, while the variational method exploited the problem's energy structure to prove existence and uniqueness. Together, these methods provided a robust framework for analyzing complex PDEs, ensuring the existence of weak solutions $Z \in W_0^{1,\beta(\cdot)}(\mathcal{A})$ and their uniqueness through the strict monotonicity of the operator. This approach is widely applicable to problems in electrorheological fluids, image processing, and materials science, though it relies on technical assumptions like the logarithm Hölder continuity condition (2) of $\alpha(\cdot)$.

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Declarations

The author declares that he has no conflicts of interest.

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