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Real and complex interpolation of Herz-type Besov-Triebel-Lizorkin spaces

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Abstract

The aim of this paper is twofold. Firstly, we study the real interpolation of Herz-type Besov-Triebel-Lizorkin spaces. Secondly, we present the complex interpolation of Herz-type Besov spaces. As application we give a simple alternative proof of Sobolev embeddings in Herz-type Triebel-Lizorkin spaces $\dot{K}_p^{\alpha,q}F_\beta^s$, $s\in\mathbb{R}, 1<p,q<\infty, 1<\beta\leq\infty$ *and* $\alpha_2\geq\alpha_1$. *These spaces unify and generalize classical Lebesgue spaces of power weights, Sobolev spaces of power weights, Besov spaces and Triebel-Lizorkin spaces.*

Keywords

Complex interpolation, Lorentz space, Real interpolation, Herz space, Herz-type Triebel-Lizorkin space, Herz-type Besov space, Maximal inequalities, Sobolev embeddings.

2020 Mathematics Subject Classification

Primary 46B70 · Secondary 46E35 ·

1. Introduction

The theory of interpolation had a remarkable development due to its usefulness in applications in mathematical analysis. For general literature on real and complex interpolation we refer to [3], [32] and references therein. Let us recall briefly the results of complex and real interpolation of some known function spaces. For Lebesgue space we have

$$[L_{p_0},L_{p_1}]_\theta=L^p,\quad 1\leq p_0,p_1<\infty,$$

and

$$(L_{p_0},L_{p_1})_{\theta,q}=L^{p,q},\quad 1\leq p_0\neq p_1\leq\infty,\quad 1\leq q\leq\infty,$$

see [3, Theorem 5.1.1], where $L^{p,q}$ is the Lorentz spaces and

$$0<\theta<1\quad\text{and}\quad\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}. \tag{1}$$

Let $L_p(\mathbb{R}^n,\omega)$ denote the weighted Lebesgue space with weight ω. The weighted version is given as follows:

$$[L_{p_0}(\mathbb{R}^n,\omega_0),L_{p_1}(\mathbb{R}^n,\omega_1)]_\theta=L_p(\mathbb{R}^n,\omega_0^\theta\omega_1^{1-\theta}),\quad 1\leq p_0,p_1<\infty, \tag{2}$$

see [3, Theorem 5.5.3] and [32, Theorem 1.18.5] with the same assumptions (1). Clearly all the above results are given for Banach case, but in [30, Lemma 3.4] the authors gave a generalization of (2) to the case $0<p_0,p_1<\infty$. For Sobolev spaces

$$[W_{p_0}^{m_0}(\mathbb{R}^n),W_{p_1}^{m_1}(\mathbb{R}^n)]_\theta=W_p^m(\mathbb{R}^n)\quad\text{and}\quad(W_{p_0}^{m_2}(\mathbb{R}^n),W_{p_1}^{m_2}(\mathbb{R}^n))_{\theta,p}=W_p^m(\mathbb{R}^n),$$

with the same assumptions (1), $1<p_0,p_1<\infty, m_0,m_1,m_2\in\mathbb{N}$ and $m=(1-\theta)m_0+\theta m_1$, see [32, Remark 2.4.2/2]. The extension of the above results to generalized scale of function spaces is given in [3] and [32].

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In this direction we present real and complex interpolation of Herz-type Besov-Triebel-Lizorkin spaces. For the problem of complex interpolation, we consider only Herz-type Besov spaces, since the complex interpolation of Herz-type Triebel-Lizorkin spaces is given in [9].

The interest in Herz-type Besov-Triebel-Lizorkin spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. In [23], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. Y. Tsutsui [35] has considered the Cauchy problem for Navier-Stokes equations on Herz spaces and weak Herz spaces. In [11] the author studied the Cauchy problem for the semilinear parabolic equations

$$\partial_t u - \Delta u = G(u)$$

with initial data in Herz-type Triebel-Lizorkin spaces and under some suitable conditions on G.

The function spaces obtained by real interpolation of Herz-type Besov and Triebel-Lizorkin spaces; see Theorem 3.25 below, are studied in detalis in [13], where the author present the φ-transform characterization of these new class of spaces in the sense of Frazier and Jawerth. In that paper one can find the Sobolev and Franke-Jewarth embeddings. Such embeddings extend and improve Sobolev, Franke and Jawerth embeddings of Besov and Triebel Lizorkin spaces. Also, the author established the smooth atomic, molecular and wavelet decomposition of these function spaces. Characterizations by ball means of differences are given.

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + ... + \alpha_n$. The Euclidean scalar product of $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ is given by $x \cdot y = x_1 y_1 + ... + x_n y_n$.

The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$.

By supp$f$ we denote the support of the function f, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. By c we denote generic positive constants, which may have different values at different occurrences.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

Its inverse is denoted by $\mathcal{F}^{-1} f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

(i) Given a measurable set $E \subset \mathbb{R}^n$ and $0 < p \leq \infty$, we denote by $L^p(E)$ the space of all functions $f : E \to \mathbb{C}$ equipped with the quasi-norm

$$\|f\|_{L^p(E)} = \Big(\int_E |f(x)|^p \, dx \Big)^{1/p} < \infty,$$

with $0 < p < \infty$ and

$$\|f\|_{L^\infty(E)} = \underset{x\in E}{\text{ess-sup}} |f(x)| < \infty.$$

If $E = \mathbb{R}^n$, then we put $L^p(\mathbb{R}^n) = L^p$ and $\|f\|_{L^p(\mathbb{R}^n)} = \|f\|_p$.

(ii) Let $\alpha \in \mathbb{R}$ and $0 < p < \infty$. The weighted Lebesgue space $L^p(\mathbb{R}^n, |\cdot|^\alpha)$ contains all measurable functions f such that

$$\|f\|_{L^p(\mathbb{R}^n,|\cdot|^\alpha)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \, |x|^\alpha dx \Big)^{1/p} < \infty.$$

If $\alpha = 0$, then we put $L^p(\mathbb{R}^n, |\cdot|^0) = L^p$.

2. The spaces $\dot{K}_p^{\alpha,q} B_\beta^s$ and $\dot{K}_p^{\alpha,q} F_\beta^s$

In this section, we present the spaces $\dot{K}_p^{\alpha,q} B_\beta^s$ and $\dot{K}_p^{\alpha,q} F_\beta^s$ on which we work and recall some of their properties. For convenience, we set

$$B_k = B(0, 2^k), \quad \bar{B}_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, \quad R_k = B_k \setminus B_{k-1} \quad \text{and} \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{Z}.$$

We begin by recalling the definition and some properties of homogeneous Herz space; see [10] and [24].

Definition 2.1. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The homogeneous Herz space $\dot{K}_p^{\alpha,q}$ is defined as the set of all $f \in L^p(\mathbb{R}^n \setminus \{0\})$ such that

$$\|f\|_{\dot{K}_p^{\alpha,q}} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f \chi_k\|_p^q \Big)^{1/q} < \infty,$$

with the usual modifications when $q = \infty$.

Remark 2.2. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$.
(i) The space $\dot{K}_p^{\alpha,p}$ coincides with the Lebesgue space $L^p(\mathbb{R}^n, |\cdot|^{\alpha p})$. In addition

$$\dot{K}_p^{0,p} = L^p.$$

(ii) Let $0 < q_1 \leq q_2 \leq \infty$. Then

$$\dot{K}_p^{\alpha,q_1} \hookrightarrow \dot{K}_p^{\alpha,q_2}.$$

(iii) The spaces $\dot{K}_p^{\alpha,q}$ are quasi-Banach spaces and if $\min(p,q) \geq 1$ then $\dot{K}_p^{\alpha,q}$ are Banach spaces.
(iv) Let $V_{\alpha,p,q}$ be the set of $(\alpha,p,q) \in \mathbb{R} \times [1,\infty]^2$ such that:
• $\alpha < n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$,
• $\alpha = n - \frac{n}{p}$, $1 \leq p \leq \infty$ and $q = 1$.Let $(\alpha,p,q) \in V_{\alpha,p,q}$. Then

$$\dot{K}_p^{\alpha,q} \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

holds.

Remark 2.3. Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [17], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [2], in the summability of Fourier transforms [15], in regularity theory for elliptic equations in divergence form [27]-[28], and in the Cauchy problem for Navier-Stokes equations [35] and in [12] for semilinear parabolic equations. But, the study of the Herz spaces can be dated back to the work of Beurling [4]. Feichtinger in [14] introduced another norm which is equivalent to the norm defined by Beurling.

Remark 2.4. A detailed discussion of the properties of Herz spaces my be found in [18], [21], [24] and [26], and references therein.

Various important results have been proved in the space $\dot{K}_p^{\alpha,q}$ under some assumptions on α, p and q. The conditions $-\frac{n}{p} < \alpha < n(1-\frac{1}{p}), 1 < p < \infty$ and $0 < q \leq \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_p^{\alpha,q}$ spaces. This fact was first realized by Li and Yang [22] with the proof of the boundedness of the maximal function were the vector valued extension is given in [31]. Let recall us recall this result. As usual, we put

$$\mathcal{M}(f)(x) = \sup_B \frac{1}{|B|} \int_B |f(y)|\, dy, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all balls of $\mathbb{R}^n$ and $x \in B$.

Lemma 2.5. *Let $1 < \beta \leq \infty, 1 < p < \infty$ and $0 < q \leq \infty$. If $\{f_j\}_{j\in\mathbb{Z}}$ is a sequence of locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$, then*

$$\Big\| \Big(\sum_{j=-\infty}^{\infty} (\mathcal{M}(f_j))^\beta \Big)^{1/\beta} \Big\|_{\dot{K}_p^{\alpha,q}} \lesssim \Big\| \Big(\sum_{j=-\infty}^{\infty} |f_j|^\beta \Big)^{1/\beta} \Big\|_{\dot{K}_p^{\alpha,q}}, \tag{3}$$

with the usual modification if $\beta = \infty$.

From Theorem 2.5 we immediately obtain the following statement.

Lemma 2.6. *Let $1 < p \leq \infty$ and $0 < q \leq \infty$. Let $f \in \dot{K}_p^{\alpha,q}$ and $-\frac{n}{p} < \alpha < n(1-\frac{1}{p})$. Then*

$$\|\mathcal{M}(f)\|_{\dot{K}_p^{\alpha,q}} \lesssim \|f\|_{\dot{K}_p^{\alpha,q}}.$$

Let A be a quasi-Banach space and $(X, \mathcal{B}, \mu)$ be a measure space with σ-finite positive measure. For $f \in L_1(A) + L_\infty(A), t > 0$ and $\lambda > 0$, we set

$$m_f(\lambda) = \mu(\{x \in X : \|f(x)\|_A > \lambda\})$$

and

$$f^*(t) = \inf\{\lambda > 0 : m_f(\lambda) \leq t\}.$$

Next we recall the definition of Lorentz spaces.

Definition 2.7. Let $0 < p < \infty$ and $0 < r \leq \infty$. Then the Lorentz space $L^{p,r}(A)$ is the set of all function $f \in L_1(A) + L_\infty(A)$, such that $\|f\|_{L^{p,r}(A)} < \infty$, where

$$\|f\|_{L^{p,r}(A)} = \Big(\int_0^\infty t^{\frac{r}{p}} (f^*(t))^r \frac{dt}{t} \Big)^{1/r} \quad \text{if} \quad 0 < r < \infty$$

and

$$\|f\|_{L^{p,\infty}(A)} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \quad \text{if} \quad r = \infty.$$

Remark 2.8. A much more detailed about Lorentz spaces can be found in [16, Chapter 1]. If $X = \mathbb{R}^n, \mu$ is the Lebesgue measure and $A = \mathbb{C}$, then we put $L^{p,r}(\mathbb{C}) = L^{p,r}$. Recall that $L^{p,p} = L^p$.

Now, we define Lorentz-Herz spaces.

Definition 2.9. Let $0 < p < \infty, 0 < q, r \leq \infty$ and $\alpha \in \mathbb{R}$. The homogeneous Herz-type Lorentz space $\dot{K}_{p,r}^{\alpha,q}$ is defined as the set of all functions f such that

$$\|f\|_{\dot{K}_{p,r}^{\alpha,q}} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f\chi_k\|_{L^{p,r}}^q \Big)^{1/q} < \infty$$

with the usual modification if $q = \infty$, i.e.,

$$\|f\|_{\dot{K}_{p,r}^{\alpha,\infty}} = \sup_{k\in\mathbb{Z}} \big(2^{k\alpha} \|f\chi_k\|_{L^{p,r}} \big).$$

Remark 2.10. Suppose $0 < q \leq \infty$. If either $0 < p, r < \infty$ or $r = \infty$ and $0 < p < \infty$, then $\dot{K}^{\alpha,q}_{p,r}$ is a quasi-Banach ideal space with the Fatou property. More detailed about Lorentz-Herz spaces is given [1].

Theorem 2.11. *Let $1 < p, \beta, r < \infty$ and $0 < q \leq \infty$. If $\{f_k\}_{k\in\mathbb{Z}}$ is a sequence of locally integrable functions on $\mathbb{R}^n$ and $-\frac{n}{p} < \alpha < n(1 - \frac{1}{p})$, then*

$$\Big\|\Big(\sum_{k=-\infty}^{\infty}(\mathcal{M}(f_k))^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,q}_{p,r}} \lesssim \Big\|\Big(\sum_{k=-\infty}^{\infty}|f_k|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,q}_{p,r}},$$

with the usual modification if $\beta = \infty$.

Proof. The proof follows easily by the same way as that the proof of Theorem 2.5, see [31], but now one has to use the Hölder's inequality for Lorentz spaces,

$$\|\chi_A\|_{L^{p,r}} = (\frac{p}{r})^{\frac{1}{r}}|A|^{\frac{1}{p}}, \quad 0 < r < \infty, \quad \|\chi_A\|_{L^{p,\infty}} = |A|^{\frac{1}{p}},$$

for any measurable set $A \subset \mathbb{R}^n$ and

$$\Big\|\Big(\sum_{k=-\infty}^{\infty}(\mathcal{M}(f_k))^{\beta}\Big)^{1/\beta}\Big\|_{L^{p,r}} \lesssim \Big\|\Big(\sum_{k=-\infty}^{\infty}|f_k|^{\beta}\Big)^{1/\beta}\Big\|_{L^{p,r}},$$

see [29, Lemma 5.1]. The proof is complete. □

Select a pair of Schwartz functions Φ and ψ such that

$$\text{supp}\mathcal{F}\Phi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\} \quad \text{and} \quad |\mathcal{F}\Phi(\xi)| \geq c > 0, \tag{4}$$

if $|\xi| \leq \frac{5}{3}$ and

$$\text{supp}\mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\} \quad \text{and} \quad |\mathcal{F}\psi(\xi)| \geq c > 0, \tag{5}$$

if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$, where $c > 0$.

Now, we define the spaces under consideration.

Definition 2.12. Let $\alpha, s \in \mathbb{R}, 0 < p, q, \beta \leq \infty, \Phi$ and ψ satisfy (4) and (5), respectively and we put $\psi_k = 2^{kn}\psi(2^k\cdot), k \in \mathbb{N}$.

(i) The Herz-type Besov space $\dot{K}^{\alpha,q}_p B^s_\beta$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}^{\alpha,q}_p B^s_\beta} = \Big(\sum_{k=0}^{\infty} 2^{ks\beta}\|\psi_k * f\|^{\beta}_{\dot{K}^{\alpha,q}_p}\Big)^{1/\beta} < \infty,$$

where ψ_0 is replaced by Φ, with the obvious modification if $\beta = \infty$.

(ii) Let $0 < p, q < \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}^{\alpha,q}_p F^s_\beta$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{K}^{\alpha,q}_p F^s_\beta} = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ks\beta}|\psi_k * f|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}^{\alpha,q}_p} < \infty,$$

where ψ_0 is replaced by Φ, with the obvious modification if $\beta = \infty$.

Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < \beta \leq \infty$. Using the system $\{\psi_k\}_{k\in\mathbb{N}_0}$ we can define the quasi-norms

$$\|f\|_{B^s_{p,\beta}} = \Big(\sum_{k=0}^{\infty} 2^{ks\beta}\|\psi_k * f\|^{\beta}_p\Big)^{1/\beta}$$

and

$$\|f\|_{F^s_{p,\beta}} = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ks\beta}|\psi_k * f|^{\beta}\Big)^{1/\beta}\Big\|_p, \quad 0 < p < \infty.$$

The Besov space $B^s_{p,\beta}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B^s_{p,\beta}} < \infty$. The Triebel-Lizorkin space $F^s_{p,\beta}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{F^s_{p,\beta}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $\{\psi_k\}_{k\in\mathbb{N}_0}$ (up to equivalence of quasi-norms). Further details on the classical theory of these spaces can be found in [33] and [34].

One recognizes immediately that if $\alpha = 0$ and $p = q$, then

$$\dot{K}^{0,p}_p B^s_\beta = B^s_{p,\beta} \quad \text{and} \quad \dot{K}^{0,p}_p F^s_\beta = F^s_{p,\beta}.$$

Moreover, for $p = q$ we re-obtain the Besov-Triebel-Lizorkin spaces of power weight; we refer, in particular, to the papers [6].

Remark 2.13. Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \leq \infty$ and $\alpha > -n$. The spaces $\dot{K}^{\alpha,q}_p B^s_\beta$ and $\dot{K}^{\alpha,q}_p F^s_\beta$ are independent of the choices of Φ and ψ. They are quasi-Banach spaces and if $p, q \geq 1$, then they are Banach spaces. Further results, concerning, for instance lifting properties, Fourier multiplier and local means characterizations can be found in [7]-[8]-[9], [36] and [37]. Some applications of such function spaces in semilinear parabolic equations are given in [11].

Let ϑ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$\vartheta(x) = 1 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \vartheta(x) = 0 \quad \text{for} \quad |x| \geq \frac{3}{2}.$$

We put $\varphi_0(x) = \vartheta(x)$, $\varphi_1(x) = \vartheta(\frac{x}{2}) - \vartheta(x)$ and $\varphi_k(x) = \varphi_1(2^{-k+1}x)$ for $k = 2, 3, ...$ Then we have supp$\varphi_k \subset \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| \leq 3 \cdot 2^{k-1}\}$ and

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{6}$$

The system of functions $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\varphi_k * f$$

for all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$). We set

$$\big\|f\big\|_{\dot{K}_p^{\alpha,q}B_\beta^s}^{\varphi_0,\varphi_1} = \Big(\sum_{k=0}^{\infty} 2^{ks\beta}\big\|\mathcal{F}^{-1}\varphi_k * f\big\|_{\dot{K}_p^{\alpha,q}}^{\beta}\Big)^{1/\beta}$$

and

$$\big\|f\big\|_{\dot{K}_p^{\alpha,q}F_\beta^s}^{\varphi_0,\varphi_1} = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ks\beta}|\mathcal{F}^{-1}\varphi_k * f|^{\beta}\Big)^{1/\beta}\Big\|_{\dot{K}_p^{\alpha,q}}.$$

For simplicity, in what follows, we use $\dot{K}_p^{\alpha,q}A_\beta^s$ to denote either $\dot{K}_p^{\alpha,q}B_\beta^s$ or $\dot{K}_p^{\alpha,q}F_\beta^s$. The case $p = \infty$ and/or $q = \infty$ is excluded when $\dot{K}_p^{\alpha,q}A_\beta^s$ means $\dot{K}_p^{\alpha,q}F_\beta^s$. In [20, Theorem 3.1] the author develop the Littlewood-Paley characterization of function spaces by introducing a new class of function spaces. By Theorem 3.1 of [20] we deduce the following properties of the spaces $\dot{K}_p^{\alpha,q}A_\beta^s$.

Theorem 2.14. *Let $s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -\frac{n}{p}$. A tempered distribution f belongs to $\dot{K}_p^{\alpha,q}A_\beta^s$ if and only if*

$$\big\|f\big\|_{\dot{K}_p^{\alpha,q}A_\beta^s}^{\varphi_0,\varphi_1} < \infty.$$

Furthermore, the quasi-norms $\big\|f\big\|_{\dot{K}_p^{\alpha,q}A_\beta^s}$ and $\big\|f\big\|_{\dot{K}_p^{\alpha,q}A_\beta^s}^{\varphi_0,\varphi_1}$ are equivalent.

Remark 2.15. We refer the reader to [19], for useful results on the Littlewood–Paley decomposition of tempered distributions.

3. Real interpolation of $\dot{K}_p^{\alpha,q}A_\beta^s$

In this section we determine the real interpolation of the spaces $\dot{K}_p^{\alpha,q}A_\beta^s$. We begin with recalling some classical results on real interpolation, see e.g. [3] and [32].

Definition 3.1. Let A_0 and A_1 be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We shall say that A_0 and A_1 are compatible if there is a Hausdorff topological vector space Z such that

$$A_0 \hookrightarrow Z \quad \text{and} \quad A_1 \hookrightarrow Z,$$

with continuous embeddings.

Let A_0 and A_1 be compatible. We will say that (A_0, A_1) is a compatible couple. Then we can form their sum $A_0 + A_1$ and their intersection $A_0 \cap A_1$. The sum consists of all $f \in Z$ such that we can write

$$f = f_0 + f_1$$

for some $f_0 \in A_0$ and $f_1 \in A_1$. Then $A_0 + A_1$ is a Banach space with norm defined by

$$\big\|f\big\|_{A_0+A_1} = \inf_{f=f_0+f_1} \big(\big\|f_0\big\|_{A_0} + \big\|f_1\big\|_{A_1}\big).$$

$A_0 \cap A_1$ is a Banach space with norm defined by

$$\big\|f\big\|_{A_0\cap A_1} = \max\big(\big\|f\big\|_{A_0}, \big\|f\big\|_{A_1}\big).$$

Let (A_0, A_1) be a compatible couple. With $t > 0$ fixed, put

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \big(\big\|f_0\big\|_{A_0} + t\big\|f_1\big\|_{A_1}\big), \quad f \in A_0 + A_1,$$

is the K-functional.

Lemma 3.2. *Let A_0 and A_1 be compatible. The functionals $t \longmapsto K(t, f; A_0, A_1)$, $t > 0$ are equivalent norms in $A_0 + A_1$.*

Now, we are in position to present the definition of the spaces $(A_0, A_1)_{\theta,q}$.

Definition 3.3. Let $0 < \theta < 1$ and $1 \leq q < \infty$. Let (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{\theta,q}$ consists of all f in $A_0 + A_1$ for which the functional

$$\|f\|_{(A_0,A_1)_{\theta,q}} = \Big(\int_0^\infty t^{-\theta q}(K(t,f))^q \frac{dt}{t}\Big)^{1/q},$$

is finite.

Definition 3.4. Let $0 < \theta < 1$. Let (A_0, A_1) be a compatible couple. The space $(A_0, A_1)_{\theta,\infty}$ consists of all f in $A_0 + A_1$ for which the functional

$$\|f\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K(t,f)$$

is finite.

We present some important properties of the spaces $(A_0, A_1)_{\theta,q}$, see [3].

Theorem 3.5. *Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Let (A_0, A_1) be a compatible couple of Banach spaces. Then $(A_0, A_1)_{\theta,q}$ is quasi-Banach space and we have*

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow A_0 + A_1.$$

Definition 3.6. Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let T be a linear operator defined on $A_0 + A_1$ and taking values in $B_0 + B_1$. T is said be admissible with respect to the couples (A_0, A_1) and (B_0, B_1) if, for each $i = 1, 0$ the restriction of T to A_i maps A_i into B_i and furthermore is a bounded operator from A_i into B_i :

$$\|Tf\|_{B_i} \leq \|T\|_{L(A_i,B_i)} \|f\|_{A_i}, \quad f \in A_i, i \in \{0, 1\}.$$

The spaces $(A_0, A_1)_{\theta,q}$ have the so called interpolation property. For the proof, see [32].

Theorem 3.7. *Let $0 < \theta < 1$ and $1 \leq q \leq \infty$. Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let T be admissible with respect to the couples (A_0, A_1) and (B_0, B_1). Then*

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$$

and

$$\|Tf\|_{(B_0,B_1)_{\theta,q}} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{(A_0,A_1)_{\theta,q}}$$

for all $f \in (A_0, A_1)_{\theta,q}$.

Definition 3.8. An interpolation functor is a rule F such that
(i) F assigns to every interpolation couple (A_0, A_1) a quasinormed space $F(A_0, A_1)$ such that

$$A_0 \cap A_1 \hookrightarrow F(A_0, A_1) \hookrightarrow A_0 + A_1.$$

(ii) If (A_0, A_1) and (B_0, B_1) are any two compatible couples and T is linear operator defined on $A_0 + A_1$ and taking values in $B_0 + B_1$ which is admissible with respect to the couples (A_0, A_1) and (B_0, B_1), then T is bounded from $F(A_0, A_1)$ to $F(B_0, B_1)$.

Definition 3.9. Let A and B be two Banach spaces. An operator $R \in L(A, B)$ is said to be a retraction if there exists an operator $S \in L(B, A)$ such that

$$R \circ S = I \quad \text{(Identity operator in } L(B, B)),$$

holds. S is called a coretraction for R.

Theorem 3.10. *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples of Banach spaces, and assume that R is linear operator defined on $A_0 + A_1$ and taking values in $B_0 + B_1$ and S is linear operator defined on $B_0 + B_1$ taking values in $A_0 + A_1$ such that R is a retraction $A_j \longrightarrow B_j$ with coretraction $S : B_j \longrightarrow A_j, j \in \{0, 1\}$. If F is any interpolation functor, then S is a coretraction $F(B_0, B_1) \longrightarrow F(A_0, A_1)$ which is an isomorphism onto a closed subspace of $F(A_0, A_1)$.*

For the proof we refer to [32].
As an immediate consequence of Theorem 3.10, we have the following theorem.

Theorem 3.11. *$0 < \theta < 1$ and $1 \leq q \leq \infty$. Let (A_0, A_1) and (B_0, B_1) be two interpolation couples of Banach spaces, and assume that R is linear operator defined on $A_0 + A_1$ and taking values in $B_0 + B_1$ and S is linear operator defined on $B_0 + B_1$ taking values in $A_0 + A_1$ such that R is a retraction $A_j \longrightarrow B_j$ with coretraction $S : B_j \longrightarrow A_j, j \in \{0, 1\}$. We have*

$$\|f\|_{(B_0,B_1)_{\theta,q}} \approx \|Sf\|_{(A_0,A_1)_{\theta,q}}$$

for any $f \in (B_0, B_1)_{\theta,q}$.

Proof. Let $f \in (B_0, B_1)_{\theta,q}$. From Theorem 3.10, we obtain

$$\|S(f)\|_{(A_0,A_1)_{\theta,q}} \lesssim \|f\|_{(B_0,B_1)_{\theta,q}}.$$

Since $(A_0, A_1)_{\theta,q}$ is interpolation functor, we have

$$\|f\|_{(B_0,B_1)_{\theta,q}} = \|R(S(f))\|_{(B_0,B_1)_{\theta,q}} \lesssim \|Sf\|_{(A_0,A_1)_{\theta,q}}.$$

The proof is complete. □

To determine the real interpolation of the spaces $\dot{K}_p^{\alpha,q} A_\beta^s$ we need some definitions and technical results.

Definition 3.12. Let A be a quasi-Banach spaces, $0 < q \le \infty$ and $\alpha \in \mathbb{R}$. Then

$$\dot{K}^{\alpha,q}(A) = \Big\{ f : f\chi_k \in A, k \in \mathbb{Z} \text{ and } \|f\|_{\dot{K}^{\alpha,q}(A)} < \infty \Big\},$$

and

$$\|f\|_{\dot{K}^{\alpha,q}(A)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f\chi_k\|_A^q \Big)^{1/q}$$

with the usual modification if $q = \infty$.

Definition 3.13. Let A be a quasi-Banach spaces, $0 < q \le \infty$ and $\alpha \in \mathbb{R}$. Then

$$\dot{\ell}_q^\alpha(A) = \Big\{ \{f_k\}_{k\in\mathbb{Z}} : f_k \in A, k \in \mathbb{Z} \text{ and } \|f\|_{\dot{\ell}_q^\alpha(A)} < \infty \Big\},$$

and

$$\|f\|_{\dot{\ell}_q^\alpha(A)} = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f_k\|_A^q \Big)^{1/q}$$

with the usual modification if $q = \infty$.

Remark 3.14. Let A be a quasi-Banach spaces, $0 < q, p \le \infty$ and $\alpha \in \mathbb{R}$. We put

$$\dot{K}_p^{\alpha,q}(A) = \dot{K}^{\alpha,q}(L^p(A)) \quad \text{and} \quad \dot{K}_p^{0,p}(A) = L^p(A).$$

We now recall some interpolation results for $\dot{\ell}_q^\alpha(A)$ and $L^p(A)$.

Theorem 3.15. *Let A, A_0 and A_1 be three Banach spaces, $0 < \theta < 1$ and $1 \le q_0, q_1, q \le \infty$.*
(i) *If $\alpha_0 \neq \alpha_1$, then*

$$(\dot{\ell}_{q_0}^{\alpha_0}(A), \dot{\ell}_{q_1}^{\alpha_1}(A))_{\theta,q} = \dot{\ell}_q^\alpha(A),$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.
(ii) *If $1 \le q_0, q_1 < \infty$, then*

$$(\dot{\ell}_{q_0}^{\alpha}(A), \dot{\ell}_{q_1}^{\alpha}(A))_{\theta,q} = \dot{\ell}_q^\alpha(A)$$

provide that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
(iii) *If $1 \le q_0, q_1 < \infty$, then*

$$(\dot{\ell}_{q_0}^{\alpha_0}(A_0), \dot{\ell}_{q_1}^{\alpha_1}(A_1))_{\theta,q} = \dot{\ell}_q^\alpha((A_0, A_1)_{\theta,q})$$

provide that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. For the proof, see [3, Theorems 5.6.1-5.6.2] and [32, Theorem 1.18.2]. □

As an important tool of this section, we need the following theorem.

Theorem 3.16. *Let A_0 and A_1 be two Banach ideal spaces, $0 < \theta < 1$. Let $1 \le q_0, q_1 \le \infty$ and $\alpha_0, \alpha_1 \in \mathbb{R}$.*
(i) *If $\alpha_0 \neq \alpha_1$, then*

$$(\dot{K}^{\alpha_0,q_0}(A), \dot{K}^{\alpha_1,q_1}(A))_{\theta,q} = \dot{K}^{\alpha,q}(A), \tag{7}$$

where $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.
(ii) *If $\alpha_0 = \alpha_1 = \alpha$ and $1 \le q_0, q_1 < \infty$, then*

$$(\dot{K}^{\alpha,q_0}(A), \dot{K}^{\alpha,q_1}(A))_{\theta,q} = \dot{K}^{\alpha,q}(A), \tag{8}$$

provide that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
(iii) *If $1 \le q_0, q_1 < \infty$, then*

$$(\dot{K}^{\alpha_0,q_0}(A_0), \dot{K}^{\alpha_1,q_1}(A_1))_{\theta,q} = \dot{K}^{\alpha,q}((A_0, A_1)_{\theta,q}), \tag{9}$$

provide that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
(iv) *If $1 \le p_0, p_1 \le \infty$ and $1 \le q_0, q_1 < \infty$, then*

$$(\dot{K}_{p_0}^{\alpha_0,q_0}(A_0), \dot{K}_{p_1}^{\alpha_1,q_1}(A_1))_{\theta,q} = \dot{K}^{\alpha,q}((L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}),$$

provide that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. Write (formally)

$$S(f) = \{f\chi_k\}_{k\in\mathbb{Z}}$$

and

$$R(\{f_k\}_{k\in\mathbb{Z}}) = \sum_{k=-\infty}^{\infty} f_k\chi_k.$$

It is obvious that the restriction of S is a linear and bounded operator

$$S : \dot{K}^{\alpha,q}(A) \to \dot{\ell}_q^{\alpha}(A).$$

Indeed, we have

$$\begin{aligned}\|S(f)\|_{\dot{\ell}_q^{\alpha}(A)} &= \|\{f\chi_k\}_{k\in\mathbb{Z}}\|_{\dot{\ell}_q^{\alpha}(A)}\\ &= \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|f\chi_k\|_A^q\Big)^{1/q}\\ &= \|f\|_{\dot{K}^{\alpha,q}(A)}\end{aligned}$$

for any $f \in \dot{K}^{\alpha,q}(A)$. Moreover, it is clear by definition of the $\chi_k, k\in\mathbb{Z}$ that

$$R(S(f)) = f \quad \text{for all} \quad f \in \dot{K}^{\alpha,q}(A).$$

Let $f = \{f_k\}_{k\in\mathbb{Z}} \in \dot{\ell}_q^{\alpha}(A)$. We have

$$\begin{aligned}\|R(f)\|_{\dot{K}^{\alpha,q}(A)} &= \|R(\{f_k\}_{k\in\mathbb{Z}})\|_{\dot{K}^{\alpha,q}(A)}\\ &= \|\sum_{k=-\infty}^{\infty} f_k\chi_k\|_{\dot{K}^{\alpha,q}(A)}\\ &= \Big(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\|(\sum_{k=0}^{\infty} f_k\chi_k)\chi_j\|_A^q\Big)^{1/q}\\ &= \Big(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\|f_j\chi_j\|_A^q\Big)^{1/q}\\ &\le \Big(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\|f_j\|_A^q\Big)^{1/q}\\ &= \|f\|_{\dot{\ell}_q^{\alpha}(A)}.\end{aligned}$$

Consequently, $\dot{K}^{\alpha,q}(A)$ is a retreat of $\ell_q^{\alpha}(A)$. Consequently, by Theorem 3.11, we find that

$$\|f\|_{(\dot{K}^{\alpha_0,q_0}(A_0),\dot{K}^{\alpha_1,q_1}(A_1))_{\theta,q}} \approx \|Sf\|_{(\dot{\ell}_{q_0}^{\alpha_0}(A_0),\dot{\ell}_{q_1}^{\alpha_1}(A_1))_{\theta,q}}$$

for any $f \in (\dot{K}^{\alpha_0,q_0}(A_0), \dot{K}^{\alpha_1,q_1}(A_1))_{\theta,q}$. Now, in case $\alpha_0 \neq \alpha_1$, the formula (7) follows from Theorem 3.15/(i). The formula (8) follows again from Theorem 3.15/(ii). Finally, the formula (9) follows by Theorem 3.15/(iii). Now, (iv) follows by (iii), since

$$\dot{K}_p^{\alpha,q}(A) = \dot{K}^{\alpha,q}(L^p(A)).$$

The proof is complete. □

Remark 3.17. We would like to mention that Theorem 3.16 is a slight variant of [1, Theorem 3.1.1.7] and [21, Theorem 3.1].

Theorem 3.18. *Let A_0 and A_1 be two Banach spaces. Let $0<\theta<1, 1\le q\le\infty, 1\le p_0,p_1\le\infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then*

$$(L^{p_0}(A_0), L^{p_1}(A_0))_{\theta,q} = L^{p,q}(A_0), \quad 1\le p_0 \neq p_1 \le \infty$$

and

$$(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,p} = L^{p}((A_0,A_1)_{\theta,p}), \quad 1\le p_0, p_1 < \infty$$

Proof. For the proof, see [32, Theorem 1.18.6/(2) and Theorem 1.18.4]. □

As an immediate consequence of Theorems 3.16 and 3.18, we have the following conclusion.

Corollary 3.19. *Let $0<\theta<1, 1\le q_0,q_1,q\le\infty$ and $\alpha_0,\alpha_1\in\mathbb{R}$.*
(i) *If $\alpha_0\neq\alpha_1$, then*

$$(\dot{K}_p^{\alpha_0,q_0}, \dot{K}_p^{\alpha_1,q_1})_{\theta,q} = \dot{K}_p^{\alpha,q},$$

where $1\le p\le\infty$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$.
(ii) *If $\alpha_0=\alpha_1=\alpha$ and $1\le q_0,q_1<\infty$, then*

$$(\dot{K}_p^{\alpha,q_0}, \dot{K}_p^{\alpha,q_1})_{\theta,q} = \dot{K}_p^{\alpha,q},$$

provide that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.
(iii) *If* $1 \le p_0 \neq p_1 \le \infty$ *and* $1 \le q_0, q_1 < \infty$, *then*

$$(\dot{K}^{\alpha_0,q_0}_{p_0}, \dot{K}^{\alpha_1,q_1}_{p_1})_{\theta,q} = \dot{K}^{\alpha,q}(L^{p,q}),$$

provide that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

In particular

$$(\dot{K}^{\alpha_0,q_0}_{p_0}, \dot{K}^{\alpha_1,q_1}_{p_1})_{\theta,q} = \dot{K}^{\alpha,q}_{q},$$

provide that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ *and* $\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

The first main result of this section is the following theorem.

Theorem 3.20. *Let* $0 < \theta < 1$. *Let* $1 < p \le \infty, 1 \le q_0, q_1 \le \infty, 1 \le \beta_0, \beta_1, \beta \le \infty$ *and* $\alpha_0, \alpha_1, s_0, s_1 \in \mathbb{R}$, *with*

$$-\frac{n}{p_0} < \alpha_0 < n - \frac{n}{p_0} \quad \text{and} \quad -\frac{n}{p_1} < \alpha_1 < n - \frac{n}{p_1}.$$

(i) *Let* $\frac{1}{q} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}, 1 \le \beta_0, \beta_1 < \infty, \alpha_0 \neq \alpha_1$

$$s = (1-\theta)s_0 + \theta s_1 \quad \text{and} \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1.$$

Then

$$(\dot{K}^{\alpha_0,q_0}_{p} B^{s_0}_{\beta_0}, \dot{K}^{\alpha_1,q_1}_{p} B^{s_1}_{\beta_1})_{\theta,q} = \dot{K}^{\alpha,q}_{p} B^{s}_{q}$$

hold in the sense of equivalent norms.
(ii) *Let* $-\frac{n}{p} < \alpha < n - \frac{n}{p}, s_0 \neq s_1$ *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$(\dot{K}^{\alpha,q}_{p} B^{s_0}_{\beta_0}, \dot{K}^{\alpha,q}_{p} B^{s_1}_{\beta_1})_{\theta,\beta} = \dot{K}^{\alpha,q}_{p} B^{s}_{\beta} \tag{10}$$

hold in the sense of equivalent norms.
(iii) *Let* $1 \le q_0, q_1, \beta_0, \beta_1 < \infty < \infty, s = (1-\theta)s_0 + \theta s_1$,

$$\frac{1}{q} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{11}$$

Then

$$(\dot{K}^{\alpha,q_0}_{p} B^{s_0}_{\beta_0}, \dot{K}^{\alpha,q_1}_{p} B^{s_1}_{\beta_1})_{\theta,q} = \dot{K}^{\alpha,q}_{p} B^{s}_{q}$$

hold in the sense of equivalent norms.
(iv) *Let* $1 \le q_0, q_1, \beta_0, \beta_1 < \infty$ *and* (11). *Then*

$$(\dot{K}^{\alpha,q_0}_{p} B^{s}_{\beta_0}, \dot{K}^{\alpha,q_1}_{p} B^{s}_{\beta_1})_{\theta,q} = \dot{K}^{\alpha,q}_{p} B^{s}_{q}$$

(v) *Let* $\alpha_0 \neq \alpha_1, 1 \le \beta_0, \beta_1 < \infty, \alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ *and* $\frac{1}{q} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$. *Then*

$$(\dot{K}^{\alpha_0,q_0}_{p} B^{s}_{\beta_0}, \dot{K}^{\alpha_1,q_1}_{p} B^{s}_{\beta_1})_{\theta,q} = \dot{K}^{\alpha,q}_{p} B^{s}_{q}$$

hold in the sense of equivalent norms.

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}_0}$ denote the smooth dyadic decomposition of unity, see (6). We define an associated system $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}_0}$ by

$$\tilde{\varphi}_0 = \varphi_0 + \varphi_1 \quad \text{and} \quad \tilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}, \quad k \in \mathbb{N}_0.$$

By using the properties of $\tilde{\varphi}_k$ we have

$$\tilde{\varphi}_k \varphi_k = \varphi_k, \quad k \in \mathbb{N}_0. \tag{12}$$

For $f \in \mathcal{S}'(\mathbb{R}^n)$ we put

$$S(f) = \{\mathcal{F}^{-1}\varphi_k * f\}_{k\in\mathbb{N}_0}.$$

For a sequence $\{g_k\}_{k\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n)$ we define (formally)

$$R(\{g_k\}_{k\in\mathbb{N}_0}) = \sum_{k=0}^{\infty} \mathcal{F}^{-1}\tilde{\varphi}_k * g_k \quad (\text{convergence in } \mathcal{S}'(\mathbb{R}^n)).$$

It is obvious that the restriction of S is a linear and bounded operator from $\dot{K}^{\alpha,q}_{p} B^{s}_{\beta}$ to $\ell^{s}_{\beta}(\dot{K}^{\alpha,q}_{p})$. Let $k \in \mathbb{N}_0, 1 < p \le \infty, 1 \le q \le \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Again, by using the properties of $\tilde{\varphi}_k$ and Lemma 2.6 we have

$$\begin{aligned}\big\|\mathcal{F}^{-1}\tilde{\varphi}_k * \mathcal{F}^{-1}\varphi_k * g_k\big\|_{\dot{K}^{\alpha,q}_{p}} &\lesssim \big\|\mathcal{M}(g_k)\big\|_{\dot{K}^{\alpha,q}_{p}} \\ &\lesssim \big\|g_k\big\|_{\dot{K}^{\alpha,q}_{p}},\end{aligned}$$

whenever $\{g_k\}_{k\in\mathbb{N}_0} \subset \dot{K}_p^{\alpha,q}$. This together with Remark 2.2 yields that R is a linear and bounded operator from $\ell_\beta^s(\dot{K}_p^{\alpha,q})$ to $\dot{K}_p^{\alpha,q}B_\beta^s$. Moreover, it is clear by (12) that

$$R(S(f)) = f \quad \text{for all} \quad f \in \dot{K}_p^{\alpha,q}B_\beta^s.$$

Consequently, the operators S and R have the properties:

- $S \in L(\dot{K}_p^{\alpha_i,q_i}B_{\beta_i}^{s_i}, \ell_{\beta_i}^{s_i}(\dot{K}_p^{\alpha_i,q_i}))$, $i \in \{0,1\}$
- $R \in L(\ell_{\beta_i}^{s_i}(\dot{K}_p^{\alpha_i,q_i}), \dot{K}_p^{\alpha_i,q_i}B_{\beta_i}^{s_i})$, $i \in \{0,1\}$
- $R(S(f)) = f$ for all $f \in \dot{K}_p^{\alpha_i,q_i}B_{\beta_i}^{s_i}$, $i \in \{0,1\}$.

Therefore, by Theorem 3.11, we obtain

$$\big\|f\big\|_{(\dot{K}_p^{\alpha_0,q_0}B_{\beta_0}^{s_0},\dot{K}_p^{\alpha_1,q_1}B_{\beta_1}^{s_1})_{\theta,q}} \approx \big\|Sf\big\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_p^{\alpha_0,q_0}),\ell_{\beta_1}^{s_1}(\dot{K}_p^{\alpha_1,q_1}))_{\theta,q}}$$

for any $f \in (\dot{K}_p^{\alpha_0,q_0}B_{\beta_0}^{s_0}, \dot{K}_p^{\alpha_1,q_1}B_{\beta_1}^{s_1})_{\theta,q}$. Theorem 3.15/(iii) and Corollary 3.19/(i) yield

$$\begin{aligned}\big\|Sf\big\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_p^{\alpha_0,q_0}),\ell_{\beta_1}^{s_1}(\dot{K}_p^{\alpha_1,q_1}))_{\theta,q}} &\approx \big\|Sf\big\|_{\ell_q^s((\dot{K}_p^{\alpha_0,q_0},\dot{K}_p^{\alpha_1,q_1})_{\theta,q})}\\ &\approx \big\|Sf\big\|_{\ell_q^s(\dot{K}_p^{\alpha,q})}\\ &\approx \big\|f\big\|_{\dot{K}_p^{\alpha,q}B_q^s}.\end{aligned}$$

This prove (i). By the same arguments as used in proof of (i) we obtain

$$\big\|f\big\|_{(\dot{K}_p^{\alpha,q}B_{\beta_0}^{s_0},\dot{K}_p^{\alpha,q}B_{\beta_1}^{s_1})_{\theta,\beta}} \approx \big\|Sf\big\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_p^{\alpha,q}),\ell_{\beta_1}^{s_1}(\dot{K}_p^{\alpha,q}))_{\theta,\beta}}$$

for any $f \in (\dot{K}_p^{\alpha,q}B_{\beta_0}^{s_0}, \dot{K}_p^{\alpha,q}B_{\beta_1}^{s_1})_{\theta,\beta}$. Theorems 3.15/(i) yields

$$\begin{aligned}\big\|Sf\big\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_p^{\alpha,q}),\ell_{\beta_1}^{s_1}(\dot{K}_p^{\alpha,q}))_{\theta,\beta}} &\approx \big\|Sf\big\|_{\ell_\beta^s(\dot{K}_p^{\alpha,q})}\\ &\approx \big\|f\big\|_{\dot{K}_p^{\alpha,q}B_\beta^s}.\end{aligned}$$

For (iii)-(v), we repeat the arguments used in (i)-(ii). This completes the proof of the theorem. □

Remark 3.21. Using the same arguments of [33, Theorem 2.4.2], (10) can be extended to $0 < p \le \infty, 0 < q_0, q_1 \le \infty$ and $0 < \beta_0, \beta_1, \beta \le \infty$

For the Herz-type Triebel-Lizorkin spaces we obtain the following statement.

Theorem 3.22. *Let* $0 < \theta < 1$. *Let* $1 < p_0, p_1, p < \infty, 1 \le q, q_0, q_1 \le \infty, 1 < \beta_0, \beta_1, \beta \le \infty$ *and* $\alpha_0, \alpha_1, s_0, s_1, s \in \mathbb{R}$, *with*

$$-\frac{n}{p_0} < \alpha_0 < n - \frac{n}{p_0} \quad \textit{and} \quad -\frac{n}{p_1} < \alpha_1 < n - \frac{n}{p_1}.$$

(i) *Let* $\alpha_0 \ne \alpha_1$ *and*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1. \tag{13}$$

Then

$$(\dot{K}_p^{\alpha_0,q_0}F_\beta^s, \dot{K}_p^{\alpha_1,q_1}F_\beta^s)_{\theta,q} = \dot{K}_p^{\alpha,q}F_\beta^s$$

hold in the sense of equivalent norms.

(ii) *Let* $-\frac{n}{p} < \alpha < n - \frac{n}{p}$,

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{14}$$

Then

$$(\dot{K}_p^{\alpha,q_0}F_\beta^s, \dot{K}_p^{\alpha,q_1}F_\beta^s)_{\theta,q} = \dot{K}_p^{\alpha,q}F_\beta^s$$

hold in the sense of equivalent norms.

(iii) *Let* $1 \le q_0, q_1 < \infty, s_0 \ne s_1, s = (1-\theta)s_0 + \theta s_1$,

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and (13)-(14). *Then*

$$(\dot{K}_{p_0}^{\alpha_0,q_0}F_{\beta_0}^{s_0}, \dot{K}_{p_1}^{\alpha_1,q_1}F_{\beta_1}^{s_1})_{\theta,q} = \dot{K}_q^{\alpha,q}F_q^s$$

hold in the sense of equivalent norms.

(iv) *Let* $1 \le \beta_0, \beta_1 < \infty$,

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$$

and (13)-(14). *Then*

$$(\dot{K}_{p_0}^{\alpha_0,q_0}F_{\beta_0}^{s}, \dot{K}_{p_1}^{\alpha_1,q_1}F_{\beta_1}^{s})_{\theta,q} = \dot{K}_q^{\alpha,q}F_q^s$$

hold in the sense of equivalent norms.

Proof. Let R and S be as in the proof of Theorem 3.20. It is clear that S is a linear and bounded operator from $\dot{K}_p^{\alpha,q}F_\beta^s$ to $\dot{K}^{\alpha,q}(L^p(\ell_\beta^s))$. Let $k \in \mathbb{N}_0, 1 < p < \infty, 1 \le \beta \le \infty, 1 \le q \le \infty$ and $-\frac{n}{p} < \alpha < n - \frac{n}{p}$. Recall that

$$|\mathcal{F}^{-1}\tilde{\varphi}_k * \mathcal{F}^{-1}\varphi_k * g_k| \lesssim \mathcal{M}(g_k),$$

whenever $\{g_k\}_{k\in\mathbb{N}_0} \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Lemma 2.5 yields R is a linear and bounded operator from $\dot{K}^{\alpha,q}(L^p(\ell_\beta^s))$ to $\dot{K}_p^{\alpha,q}F_\beta^s$. Again, by (12) we have that

$$R(S(f)) = f \quad \text{for all} \quad f \in \dot{K}_p^{\alpha,q}F_\beta^s.$$

Consequently, the operators S and R have the properties:

- $S \in L(\dot{K}_p^{\alpha_i,q_i}F_\beta^s, \dot{K}^{\alpha_i,q_i}(L^p(\ell_\beta^s)))$, $i \in \{0,1\}$
- $R \in L(\dot{K}^{\alpha_i,q_i}(L^p(\ell_\beta^s)), \dot{K}_p^{\alpha_i,q_i}F_\beta^s)$, $i \in \{0,1\}$
- $R(S(f)) = f$ for all $f \in \dot{K}_p^{\alpha_i,q_i}F_\beta^s, i \in \{0,1\}$.

By Theorem 3.11, we obtain

$$\big\|f\big\|_{(\dot{K}_p^{\alpha_0,q_0}F_\beta^s,\dot{K}_p^{\alpha_1,q_1}F_\beta^s)_{\theta,q}} \approx \big\|Sf\big\|_{(\dot{K}^{\alpha_0,q_0}(L^p(\ell_\beta^s)),\dot{K}^{\alpha_1,q_1}(L^p(\ell_\beta^s))_{\theta,q}}$$

for any $f \in (\dot{K}_p^{\alpha_0,q_0}F_\beta^s, \dot{K}_p^{\alpha_1,q_1}F_\beta^s)_{\theta,q}$. Theorem 3.16/(i) yields

$$\begin{aligned}\big\|Sf\big\|_{(\dot{K}^{\alpha_0,q_0}(L^p(\ell_\beta^s)),\dot{K}^{\alpha_1,q_1}(L^p(\ell_\beta^s))_{\theta,q}} &\approx \big\|Sf\big\|_{\dot{K}_p^{\alpha,q}(\ell_\beta^s)}\\ &\approx \big\|f\big\|_{\dot{K}_p^{\alpha,q}F_\beta^s}.\end{aligned}$$

This proves (i). For (ii), we get

$$\big\|f\big\|_{(\dot{K}_p^{\alpha,q_0}F_\beta^s,\dot{K}_p^{\alpha,q_1}F_\beta^s)_{\theta,q}} \approx \big\|Sf\big\|_{(\dot{K}^{\alpha,q_0}(L^p(\ell_\beta^s)),\dot{K}^{\alpha,q_1}(L^p(\ell_\beta^s))_{\theta,q}} \approx \big\|f\big\|_{\dot{K}_p^{\alpha,q}F_\beta^s}$$

for any $f \in (\dot{K}_p^{\alpha,q_0}F_\beta^s, \dot{K}_p^{\alpha,q_1}F_\beta^s)_{\theta,q}$, where we used Theorem 3.16/(ii). Similarly, we obtain

$$\begin{aligned}\big\|f\big\|_{(\dot{K}_{p_0}^{\alpha_0,q_0}F_{\beta_0}^{s_0},\dot{K}_{p_1}^{\alpha_1,q_1}F_{\beta_1}^{s_1})_{\theta,q}} &\approx \big\|Sf\big\|_{(\dot{K}^{\alpha_0,q_0}(L^{p_0}(\ell_{\beta_0}^{s_0})),\dot{K}^{\alpha_1,q_1}(L^{p_1}(\ell_{\beta_1}^{s_1}))_{\theta,q}}\\ &\approx \big\|Sf\big\|_{\dot{K}^{\alpha,q}((L^{p_0}(\ell_{\beta_0}^{s_0}),L^{p_1}(\ell_{\beta_1}^{s_1}))_{\theta,q})}\\ &\approx \big\|Sf\big\|_{\dot{K}^{\alpha,q}(L^q((\ell_{\beta_0}^{s_0},\ell_{\beta_1}^{s_1})_{\theta,q}))}\\ &\approx \big\|Sf\big\|_{\dot{K}_q^{\alpha,q}(\ell_q^s)}\\ &\approx \big\|f\big\|_{\dot{K}_q^{\alpha,q}F_q^s}\end{aligned}$$

for any $f \in (\dot{K}_{p_0}^{\alpha_0,q_0}F_{\beta_0}^{s_0}, \dot{K}_{p_1}^{\alpha_1,q_1}F_{\beta_1}^{s_1})_{\theta,q}$. The proof of (iv) follows by the same arguments. □

Before going to formulate more general real interpolation of Herz-Besov spaces, we need some preparation.

Definition 3.23. Let $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be a resolutions of unity. Let $s, \alpha \in \mathbb{R}, 0 < p < \infty, 0 < r \le \infty$ and $0 < \beta, q \le \infty$.
(i) The Herz-type Lorentz Besov space $\dot{K}_{p,r}^{\alpha,q}B_\beta^s$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{K}_{p,r}^{\alpha,q}B_\beta^s} = \Big(\sum_{k=0}^{\infty} 2^{ks\beta}\big\|\mathcal{F}^{-1}\varphi_k * f\big\|_{\dot{K}_{p,r}^{\alpha,q}}^{\beta}\Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.
(ii) Let $0 < p, q, r < \infty$. The Herz-type Lorentz Triebel-Lizorkin space $\dot{K}_{p,r}^{\alpha,q}F_\beta^s$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f\big\|_{\dot{K}_{p,r}^{\alpha,q}F_\beta^s} = \Big\|\Big(\sum_{k=0}^{\infty} 2^{ks\beta}|\mathcal{F}^{-1}\varphi_k * f|^\beta\Big)^{1/\beta}\Big\|_{\dot{K}_{p,r}^{\alpha,q}} < \infty,$$

with the obvious modification if $\beta = \infty$.

For simplicity, in what follows, we use $\dot{K}_{p,r}^{\alpha,q}A_\beta^s$ to denote either $\dot{K}_{p,r}^{\alpha,q}B_\beta^s$ or $\dot{K}_{p,r}^{\alpha,q}F_\beta^s$.

Theorem 3.24. *Let $s \in \mathbb{R}, 0 < p < \infty, 0 < \beta, r, q \le \infty$ and $\alpha > -\frac{n}{p}$. The definition of the spaces $\dot{K}_{p,r}^{\alpha,q}A_\beta^s$ is independent of the choices of the partition of unity $\{\varphi_k\}_{k\in\mathbb{N}_0}$.*

Proof. The proof follows by Theorem 2.11. □

Now we are in position to prove the following result.

Theorem 3.25. *Let $0 < \theta < 1$. Let $1 \le p_0 \ne p_1 \le \infty, 1 \le q_0, q_1 < \infty, 1 \le \beta_0, \beta_1, \beta \le \infty$ and $\alpha_0, \alpha_1, s_0, s_1 \in \mathbb{R}$, with*

$$-\frac{n}{p_0} < \alpha_0 < n - \frac{n}{p_0} \quad \text{and} \quad -\frac{n}{p_1} < \alpha_1 < n - \frac{n}{p_1}.$$

Assume that

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

(i) *We have*

$$(\dot{K}_{p_0}^{\alpha_0,q_0} F_\beta^s, \dot{K}_{p_1}^{\alpha_1,q_1} F_\beta^s)_{\theta,q} = \dot{K}_{p,q}^{\alpha,q} F_\beta^s$$

hold in the sense of equivalent norms.
(ii) *Let $\frac{1}{q} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$(\dot{K}_{p_0}^{\alpha_0,q_0} B_{\beta_0}^{s_0}, \dot{K}_{p_1}^{\alpha_1,q_1} B_{\beta_1}^{s_1})_{\theta,q} = \dot{K}_{p,q}^{\alpha,q} B_q^s$$

hold in the sense of equivalent norms.

Proof. Let R and S be as in the proof of Theorem 3.20. By Theorem 3.11, we obtain

$$\|f\|_{(\dot{K}_{p_0}^{\alpha_0,q_0} F_\beta^s, \dot{K}_{p_0}^{\alpha_1,q_1} F_\beta^s)_{\theta,q}} \approx \|Sf\|_{(\dot{K}^{\alpha_0,q_0}(L^{p_0}(\ell_\beta^s)), \dot{K}^{\alpha_1,q_1}(L^{p_1}(\ell_\beta^s))_{\theta,q}}$$

for any $f \in (\dot{K}_{p_0}^{\alpha_0,q_0} F_\beta^s, \dot{K}_{p_0}^{\alpha_1,q_1} F_\beta^s)_{\theta,q}$. From Theorems 3.16 and 3.18, we obtain

$$\begin{aligned} \|Sf\|_{(\dot{K}^{\alpha_0,q_0}(L^{p_0}(\ell_\beta^s)), \dot{K}^{\alpha_1,q_1}(L^{p_1}(\ell_\beta^s))_{\theta,q}} &\approx \|Sf\|_{\dot{K}^{\alpha,q}((L^{p_0}(\ell_\beta^s), L^{p_1}(\ell_\beta^s))_{\theta,q})} \\ &\approx \|Sf\|_{\dot{K}^{\alpha,q}(L^{p,q}(\ell_\beta^s))} \\ &\approx \|f\|_{\dot{K}_{p,q}^{\alpha,q} F_\beta^s}. \end{aligned}$$

This prove (i). Similarly, we obtain

$$\|f\|_{(\dot{K}_{p_0}^{\alpha_0,q_0} B_{\beta_0}^{s_0}, \dot{K}_{p_0}^{\alpha_1,q_1} B_{\beta_1}^{s_1})_{\theta,q}} \approx \|Sf\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_{p_0}^{\alpha_0,q_0}), \ell_{\beta_1}^{s_1}(\dot{K}_{p_1}^{\alpha_1,q_1}))_{\theta,q}}$$

for any $f \in (\dot{K}_{p_0}^{\alpha_0,q_0} B_{\beta_0}^{s_0}, \dot{K}_{p_0}^{\alpha_1,q_1} B_{\beta_1}^{s_1})_{\theta,q}$. By Theorems 3.15 and 3.16, we obtain

$$\begin{aligned} \|Sf\|_{(\ell_{\beta_0}^{s_0}(\dot{K}_{p_0}^{\alpha_0,q_0}), \ell_{\beta_1}^{s_1}(\dot{K}_{p_1}^{\alpha_1,q_1}))_{\theta,q}} &\approx \|Sf\|_{\ell_q^s((\dot{K}_{p_0}^{\alpha_0,q_0}, \dot{K}_{p_1}^{\alpha_1,q_1})_{\theta,q})} \\ &\approx \|Sf\|_{\ell_q^s(\dot{K}^{\alpha,q}(L^{p,q}))} \\ &\approx \|f\|_{\dot{K}_{p,q}^{\alpha,q} B_q^s}. \end{aligned}$$

This completes the proof. □

3.1. Complex interpolation of $\dot{K}_p^{\alpha,q} B_\beta^s$

In this section we establish the complex interpolation of Herz-type Besov spaces. First, we need some preparations.

Definition 3.26. Let (A_0, A_1) be an interpolation couple of quasi-Banach spaces, i.e., A_j, $j = 0, 1$, are continuously imbedded in a larger Hausdorff topological vector space, and $X_0 \cap X_1$ is dense in $A_j, j = 0, 1$. In addition, let $A_0 + A_1$ be analytically convex. Define $\mathcal{F}$ as the space of bounded analytic functions $f : A \to A_0 + A_1$, which extend continuously to the closure $\overline{A}$, such that the traces $t \to f(j + it)$ are bounded continuous functions into $A_j, j = 0, 1$. We endow $\mathcal{F}$ with the quasi-norm

$$\|f\|_{\mathcal{F}} = \max\big(\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}\big).$$

Further, we define the complex interpolation space

$$[A_0, A_1]_\theta = \{x \in A_0 + A_1 : x = f(\theta) \text{ for some } f \in \mathcal{F}\}, \quad 0 < \theta < 1$$

and

$$\|x\|_{[A_0,A_1]_\theta} = \inf\{\|f\|_X : f \in \mathcal{F}, \quad f(\theta) = x\}.$$

Remark 3.27. Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces. We have

$$A_0 \cap A_1 \hookrightarrow [A_0, A_1]_\theta \hookrightarrow A_0 + A_1,$$

see [3].

Also, the spaces $[A_0, A_1]_\theta$ have the so called interpolation property.

Theorem 3.28. *Let $0 < \theta < 1$. Let (A_0, A_1) and (B_0, B_1) be two compatible couples of Banach spaces and let T be admissible with respect to the couples (A_0, A_1) and (B_0, B_1). Then*

$$T : [A_0, A_1]_\theta \longrightarrow [B_0, B_1]_\theta$$

and

$$\|Tf\|_{[B_0,B_1]_\theta} \leq \|T\|^{1-\theta}_{L(A_0,B_0)} \|T\|^{\theta}_{L(A_1,B_1)} \|f\|_{[A_0,A_1]_\theta}$$

for all $f \in [A_0, A_1]_\theta$.

Proof. For the proof, see [3, Theorem 4.1.2]. □

As an important tool of this section, we need the following two theorems.

Theorem 3.29. $0 < \theta < 1$. *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples of Banach spaces, and assume that R is linear operator defined on $A_0 + A_1$ and taking values in $B_0 + B_1$ and S is linear operator defined on $B_0 + B_1$ taking values in $A_0 + A_1$ such that R is a retraction $A_j \longrightarrow B_j$ with coretraction $S : B_j \longrightarrow A_j, j \in \{0, 1\}$. We have*

$$\|f\|_{[B_0,B_1]_\theta} \approx \|Sf\|_{[A_0,A_1]_\theta}$$

for any $f \in [B_0, B_1]_\theta$.

Remark 3.30. Further details on the theory of interpolation can be found in [3] and [32].

Theorem 3.31. *Let A_0 and A_1 be two Banach spaces. Let $0 < \theta < 1, 1 \leq q_0 < \infty, 1 \leq q_1 \leq \infty$ and*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

(i) *Let $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$[\dot{\ell}^{\alpha_0}_{q_0}(A_0), \dot{\ell}^{\alpha_1}_{q_1}(A_1)]_\theta = \dot{\ell}^{\alpha}_{q}([A_0, A_1]_\theta).$$

(ii) *Let $1 \leq q_0, q_1 < \infty$. Then*

$$[L^{q_0}(A_0), L^{q_1}(A_1)]_\theta = L^q([A_0, A_1]_\theta).$$

Proof. For the proof, see [3] and [32, Theorems 1.18.1 and 1.18.4]. □

Based on Theorems 3.29-3.31 one derives the following.

Theorem 3.32. *Let A_0 and A_1 be two Banach ideal spaces, $0 < \theta < 1$. Let $1 \leq q_0 < \infty, 1 \leq q_1 \leq \infty$ and $\alpha_0, \alpha_1 \in \mathbb{R}$,*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

(i) *We have*

$$[\dot{K}^{\alpha_0,q_0}(A_0), \dot{K}^{\alpha_1,q_1}(A_1)]_\theta = \dot{K}^{\alpha,q}([A_0, A_1]_\theta),$$

(ii) *If $1 \leq p_0, p_1 < \infty$, then*

$$[\dot{K}^{\alpha_0,q_0}_{p_0}(A_0), \dot{K}^{\alpha_1,q_1}_{p_1}(A_1)]_\theta = \dot{K}^{\alpha,q}(L^p([A_0, A_1]_\theta).$$

provided that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. In particular

$$[\dot{K}^{\alpha_0,q_0}_{p_0}, \dot{K}^{\alpha_1,q_1}_{p_1}]_\theta = \dot{K}^{\alpha,q}_{p}.$$

Proof. Let R and S be as in the proof of Theorem 3.16. By Theorem 3.29, we find that

$$\|f\|_{[\dot{K}^{\alpha_0,q_0}(A_0),\dot{K}^{\alpha_1,q_1}(A_1)]_\theta} \approx \|Sf\|_{[\dot{\ell}^{\alpha_0}_{q_0}(A_0),\dot{\ell}^{\alpha_1}_{q_1}(A_1)]_\theta}$$

for any $f \in [\dot{K}^{\alpha_0,q_0}(A_0), \dot{K}^{\alpha_1,q_1}(A_1)]_\theta$. Using Theorem 3.31, we obtain

$$\|f\|_{[\dot{K}^{\alpha_0,q_0}(A),\dot{K}^{\alpha_1,q_1}(A)]_\theta} \approx \|Sf\|_{\dot{\ell}^{\alpha}_{q}([A_0,A_1]_\theta)} \approx \|f\|_{\dot{K}^{\alpha,q}([A_0,A_1]_\theta)}.$$

This prove (i). Now, (ii) follows by (i). The proof is complete. □

After these preparations we are ready to present the main result of this section.

Theorem 3.33. *Let $0 < \theta < 1$. Let $1 \leq p_0, p_1 < \infty, 1 \leq q_0, q_1 \leq \infty, 1 \leq \beta_0 < \infty, 1 \leq \beta_1 \leq \infty$ and $\alpha_0, \alpha_1, s_0, s_1 \in \mathbb{R}$, with $-\frac{n}{p_0} < \alpha_0 < n - \frac{n}{p_0}$ and $-\frac{n}{p_1} < \alpha_1 < n - \frac{n}{p_1}$. We put*

$$\frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1} \quad \text{and} \quad s = (1-\theta)s_0 + \theta s_1.$$

Let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then

$$[\dot{K}^{\alpha_0,q_0}_{p_0} B^{s_0}_{\beta_0}, \dot{K}^{\alpha_1,q_1}_{p_1} B^{s_1}_{\beta_1}]_\theta = \dot{K}^{\alpha,q}_{p} B^{s}_{\beta}$$

hold in the sense of equivalent norms.

Proof. The proof is similar as in Theorem 3.20, but now one has to use Theorems 3.31 and 3.32. □

4. Application

In this section we present an application of Theorem 3.22. More precisely, we present a simple alternative proof of Sobolev embeddings in Herz-type Triebel-Lizorkin spaces $\dot{K}_p^{\alpha,q}F_\beta^{s_1}$, $s\in\mathbb{R}, 1<p,q<\infty, 1<\beta\leq\infty$ and $\alpha_2\geq\alpha_1$. Such embeddings are given in [8]. We begin by the following statement which can be found in [5], that plays an essential role later on.

Lemma 4.1. *Let real numbers $s_1<s_0$ be given, and $0<\sigma<1$. For $0<q\leq\infty$ there is $c>0$ such that*

$$\Big(\sum_{j=0}^{\infty}2^{(\sigma s_0+(1-\sigma)s_1)qj}|a_j|^q\Big)^{1/q}\leq c\sup_{j\in\mathbb{N}_0}(2^{s_0j}|a_j|)^{\sigma}\sup_{j\in\mathbb{N}_0}(2^{s_1j}|a_j|)^{1-\sigma}$$

holds for all complex sequences $\left\{2^{s_0j}a_j\right\}_{j\in\mathbb{N}_0}$ in ℓ_∞ with the usual modification if $q=\infty$.

We are ready to prove the following statement.

Theorem 4.2. *Let $\alpha_1,\alpha_2,s_1,s_2\in\mathbb{R}, 1<s,r,p,q<\infty, 1\leq\beta,\theta\leq\infty, -\frac{n}{s}<\alpha_1<n-\frac{n}{s}$ and $-\frac{n}{q}<\alpha_2<n-\frac{n}{q}$. We suppose that*

$$s_1-\frac{n}{s}-\alpha_1=s_2-\frac{n}{q}-\alpha_2.$$

Let $1<q<s<\infty$ and $\alpha_2\geq\alpha_1$. Then

$$\dot{K}_q^{\alpha_2,r}F_\theta^{s_2}\hookrightarrow\dot{K}_s^{\alpha_1,p}F_\beta^{s_1}, \tag{15}$$

if and only if $1<r\leq p<\infty$.

Proof. The necessity of $1<r\leq p<\infty$ was proved in [8]. We decompose the proof into two steps.

Step 1. We will prove that

$$\dot{K}_q^{\alpha_2,q}F_\theta^{s_2}\hookrightarrow\dot{K}_s^{\alpha_1,s}F_1^{s_1}. \tag{16}$$

This embedding was proved in [25]. For the convenience of the reader we present the proof. We set

$$\sigma_0=1-\frac{\alpha_1+\frac{n}{s}}{\alpha_2+\frac{n}{q}}\quad\text{and}\quad\alpha_3=\frac{\alpha_1-(1-\sigma)\alpha_2}{\sigma},\quad\sigma_0<\sigma\leq1.$$

Obviously, $0<\sigma_0<1$ and $\alpha_1=\sigma\alpha_3+(1-\sigma)\alpha_2$. We set

$$\frac{1}{s}=\frac{1-\sigma}{q}+\frac{\sigma}{v}.$$

Note that $1<q<s<\infty$ implies $v>s$. Since $\sigma_0<\sigma$, we have $\alpha_3>-\frac{n}{v}$. Let further s_3 be defined by

$$s_3=\frac{n}{v}+\alpha_3+s_1-\frac{n}{s}-\alpha_1.$$

Observe that $s_3<s_1$. These guarantee that

$$\dot{K}_s^{\alpha_1,s}F_1^{s_1}\hookrightarrow\dot{K}_s^{\alpha_1,s}B_v^{s_1}\hookrightarrow\dot{K}_v^{\alpha_3,v}B_v^{s_3}\hookrightarrow\dot{K}_v^{\alpha_3,v}F_\infty^{s_3}.$$

Let $f\in\dot{K}_q^{\alpha_2,q}F_\theta^{s_2}$. By Lemma 4.1, $s_1=\sigma s_3+(1-\sigma)s_2$ and Hölder's inequality, we obtain

$$\begin{aligned}\|f\|_{\dot{K}_s^{\alpha_1,s}F_1^{s_1}}&\leq\|f\|^{1-\sigma}_{\dot{K}_q^{\alpha_2,q}F_\infty^{s_2}}\|f\|^{\sigma}_{\dot{K}_v^{\alpha_3,v}F_\infty^{s_3}}\\&\lesssim\|f\|^{1-\sigma}_{\dot{K}_q^{\alpha_2,q}F_\infty^{s_2}}\|f\|^{\sigma}_{\dot{K}_s^{\alpha_1,s}F_1^{s_1}}.\end{aligned}$$

This leads to (16).

Step 2. We prove (15). Since $\alpha_2<n-\frac{n}{q}$, we have $\alpha_1<s_1-s_2+n-\frac{n}{s}$. Let α_1^0,α_1^1 be such that

$$\alpha_1^0<\alpha_1<\alpha_1^1<s_1-s_2+n-\frac{n}{s}$$

and $\alpha_1^0>-\frac{n}{s}$. Write

$$\alpha_1=\sigma_1\alpha_1^0+(1-\sigma_1)\alpha_1^1,\quad 0<\sigma_1<1.$$

We put

$$\alpha_i^2=s_2-\frac{n}{q}-s_1+\frac{n}{s}+\alpha_i^1,\quad i\in\{0,1\}.$$

Since $s_1-\frac{n}{s}\leq s_2-\frac{n}{q}$, we have $\alpha_i^2\geq\alpha_i^1>-\frac{n}{s}>-\frac{n}{q}, i\in\{0,1\}$ and

$$\alpha_2=\sigma_1\alpha_2^0+(1-\sigma_1)\alpha_2^1.$$

In addition $\alpha_i^2<n-\frac{n}{q}, i\in\{0,1\}$. Then we have

$$\dot{K}_q^{\alpha_2^i,q}F_\theta^{s_2}\hookrightarrow\dot{K}_s^{\alpha_1^i,s}F_\beta^{s_1},\quad i\in\{0,1\}.$$

By interpolation, we get

$$\begin{aligned}(\dot{K}_q^{\alpha_2^0,q}F_\theta^{s_2},\dot{K}_q^{\alpha_2^1,q}F_\theta^{s_2})_{\sigma_1,r} &\hookrightarrow (\dot{K}_s^{\alpha_1^0,s}F_\beta^{s_1},\dot{K}_s^{\alpha_1^1,s}F_\beta^{s_1})_{\sigma_1,r}\\ &\hookrightarrow (\dot{K}_s^{\alpha_1^0,s}F_\beta^{s_1},\dot{K}_s^{\alpha_1^1,s}F_\beta^{s_1})_{\sigma_1,p}.\end{aligned}$$

By Theorem 3.22/(i), we get

$$\dot{K}_q^{\alpha_2,r}F_\theta^{s_2} = (\dot{K}_q^{\alpha_2^0,q}F_\theta^{s_2},\dot{K}_q^{\alpha_2^1,q}F_\theta^{s_2})_{\sigma_1,r}$$

and

$$\dot{K}_s^{\alpha_1,p}F_\beta^{s_1} = (\dot{K}_s^{\alpha_1^0,s}F_\beta^{s_1},\dot{K}_s^{\alpha_1^1,s}F_\beta^{s_1})_{\sigma_1,p}.$$

This yields the desired estimate. The proof is complete. □

Remark 4.3. The method given in Theorem 4.2 is also valid to prove some other Sobolev embeddings, see e.g. [25]. Indeed, let $F_q^s(L^{p,r})$ be the Lorentz-Triebel-Lizorkin spaces, see [32, 2.4.2]. Let $0 < p_0, p_1 < \infty$ and $0 < q_0, q_1, r_0, r_1 \leq \infty$. In [29] the authors proved that

$$F_{q_0}^{s_0}(L^{p_0,r_0}) \hookrightarrow F_{q_1}^{s_1}(L^{p_1,r_1}) \tag{17}$$

holds if and only if one of the following four conditions is satisfied.
(i) $s_0 - s_1 > \frac{n}{p_0} - \frac{n}{p_1} > 0$.
(ii) $s_0 > s_1, p_0 = p_1, r_0 \leq r_1$.
(iii) $s_0 - s_1 = \frac{n}{p_0} - \frac{n}{p_1} > 0, r_0 \leq r_1$.
(iv) $s_0 = s_1, p_0 = p_1, r_0 \leq r_1, q_0 \leq q_1$.
In the case of $1 < p_0, p_1 < \infty$ and $1 < q_0, q_1, r_0, r_1 < \infty$, we present an alternative proof of (iii). Indeed, let p_0^0, p_0^1 be such that

$$\frac{1}{p_0} = \frac{1-\sigma}{p_0^0} + \frac{\sigma}{p_0^1}, \quad 0 < \sigma < 1.$$

We set

$$\frac{n}{p_1^i} = s_1 - s_0 + \frac{n}{p_0^i}, \quad i \in \{0,1\}.$$

Then $p_0^i < p_1^i, i \in \{0,1\}$ and

$$F_{p_0^i,q_0}^{s_0} \hookrightarrow F_{p_1^i,q_1}^{s_1}, \quad i \in \{0,1\},$$

see [33]. By interpolation, we get

$$\begin{aligned}(F_{p_0^0,q_0}^{s_0},F_{p_0^1,q_0}^{s_0})_{\sigma,r_0} &\hookrightarrow (F_{p_1^0,q_1}^{s_1},F_{p_1^1,q_1}^{s_1})_{\sigma,r_0}\\ &\hookrightarrow (F_{p_1^0,q_1}^{s_1},F_{p_1^1,q_1}^{s_1})_{\sigma,r_1}.\end{aligned}$$

By [32, Theorem 2.4.2/1], we get

$$F_{q_0}^{s_0}(L^{p_0,r_0}) = (F_{p_0^0,q_0}^{s_0},F_{p_0^1,q_0}^{s_0})_{\sigma,r_0}$$

and

$$F_{q_1}^{s_1}(L^{p_1,r_1}) = (F_{p_1^0,q_1}^{s_1},F_{p_1^1,q_1}^{s_1})_{\sigma,r_1}.$$

This yields the desired embeddings (17) under the assumption $s_0 - s_1 = \frac{n}{p_0} - \frac{n}{p_1} > 0, r_0 \leq r_1$.

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Declarations

The author declares that he has no conflicts of interest.

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