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On Fixed Point Theorems for Self-Mappings in Complex Metric Spaces with Special Functions

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abstract

This paper delves into the forefront of fixed point theory, focusing on recent advancements within the context of contraction mappings in complex metric spaces. The study introduces a novel perspective by incorporating the pivotal role of control functions in elucidating the behavior and properties of fixed points. We investigate the interplay between contraction mappings and complex metric spaces using a control function. We provide an example to illustrate our findings.

keywords

Complex metric space, Special function, Generalized contraction.

2020 Mathematics Subject Classification

47H10 · 54H25 ·

1. Introduction and Preliminaries

In recent years, fixed point theory has witnessed a surge of interest and innovation, particularly in the exploration of contraction mappings within the intricate realm of complex metric spaces, see (Bhatt et al., 2011, Kang et al., 2013, Kutbi et al., 2013, Ahmad et al., 2013, Manro, 2013, Mohanta and Maitra, 2012, Rouzkard and Imdad, 2012, Sintunavarat and Kumam, 2012, Sitthikul and Saejung, 2012, Verma and Pathak, 2013). This paper aims to contribute to this evolving discourse by investigating novel perspectives and advancements in the field, with a particular focus on the pivotal role of control functions in shaping the dynamics of fixed point iterations. Firstly, in the preliminary table, we need to define a new partial order relation $\precsim$ on C.
Let C be the set of complex numbers and $z_1, z_2 \in C$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following cases is satisfied:

$$\begin{aligned} Re(z_1) = Re(z_2), Im(z_1) < Im(z_2), \\ Re(z_1) < Re(z_2), Im(z_1) = Im(z_2), \\ Re(z_1) < Re(z_2), Im(z_1) < Im(z_2), \\ Re(z_1) = Re(z_2), Im(z_1) = Im(z_2). \end{aligned}$$

we write $z_1 \precnsim z_2$ if $z_1 \precnsim z_2$ and $z_1 \neq z_2$,and we will write $z_1 \prec z_2$ if only (3) is satisfied.Note that

$$0 \precsim z1 \precnsim z2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2 \text{ and } z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$
$$0 \precsim z1 \precsim z2 \Rightarrow |z_1| \leq |z_2|.$$

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Definition 1.1. Azam Azam et al., 2011 Let X be a nonempty set. Suppose that the function $d : X \times X \to \mathbb{C}$,satisfies.
(a) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$,
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(c) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
d is called a complex valued metric in X and The pair (X, d) is called a complex valued metric space.

Example 1.2. Sintunavarat and Kumam, 2012 Let $X = \mathbb{C}$ Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d\left(z_1, z_2\right) = \exp\left(ik\right)\left|z_1 - z_2\right|^2,$$

where $k \in \left[0, \frac{\pi}{2}\right]$. Then (X, d) is a complex valued metric space.

Definition 1.3. Azam Azam et al., 2011 Suppose that (X, d) be a complex valued metric space and $\{x_n\}$ be a sequence in X and $x \in X$., We find that
(i) the sequence $\{x_n\}$ converges to $x_0 \in X$ if for every $0 < c \in \mathbb{C}$, there exists an integer N such that $d(x_n, x_0) < c$ for all $n \geq N$.
we write $x_n \to x_0$.
(ii) the sequence $\{x_n\}$ is a Cauchy sequence if for every $0 < c \in \mathbb{C}$, there exists an integer N such that $d(x_n, x_m) < c$ for all $n, m \geq N$.
(iii) the metric space (X, d) is complete, if every Cauchy sequence in X converges to a point in X.

Lemma 1.4. *Azam Azam et al., 2011 Let (X, d) be a complex valued metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converge to x_0 if and only if $|d(x_n, x_0)| \to 0$ as $n \to \infty$.*

Lemma 1.5. *Azam Azam et al., 2011 Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$,*

Lemma 1.6. *Azam Azam et al., 2011 let $\{x_n\}$ be a sequence in X and $h \in [0, 1)$ .if $a_n = |d(x_n, x_{n+1})|$ satisfies*

$$a_n \leq h a_{n-1}, \text{ for all } n \in N,$$

then $\{x_n\}$ is a Cauchy sequence.

2. Main results

Firstly, in this chapter, we will need to utilize the following assumption.
Throughout this work, Let (X, d) be a complex valued metric space and let $S, T : X \to X$.

Proposition 2.1. *Let $x_0 \in X$ and defined the sequence $\{x_n\}$ be defined by*

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \text{ for all } n = 0, 1, 2, ...$$

Assume that there exists a control function $\gamma : X \times X \to [0, 1)$ satisfying.

$$\gamma(TSx, y) \leq \gamma(x, y) \text{ and } \gamma(x, STy) \leq \gamma(x, y)$$

for all $x, y \in X$. then

$$\gamma(x_{2n}, y) \leq \gamma(x_0, y) \text{ and } \gamma(x, x_{2n+1}) \leq \gamma(x, x_1)$$

for all $x, y \in X$ and $n = 0, 1, 2, ...$

Proof. let $x, y \in X$ and $n = 0, 1, 2, ...$ then we have

$$\begin{aligned}\gamma(x_{2n}, y) &= \gamma(TSx_{2n-2}, y) \leq \gamma(x_{2n-2}, y) \\ &= \gamma(TSx_{2n-4}, y) \leq ... \leq \gamma(x_0, y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\gamma(x, x_{2n+1}) &= \gamma(x, STx_{2n-1}) \leq \gamma(x, x_{2n-1}) \\ &= \gamma(x, STx_{2n-3}) \leq ... \leq \gamma(x, x_1).\end{aligned}$$

□

Theorem 2.2. *Let (X,d) be a complex valued metric space and let $S,T : X \to X$. if there exist the control function $\gamma : X \times X \to [0,1)$ such that for all $x,y \in X$:*

(a)

$$\gamma(TSx,y) \leq \gamma(x,y) \text{ and } \gamma(x,STy) \leq \gamma(x,y);$$

(b)

$$\gamma(x_0,x_1) < 1, \tag{1}$$

(c)

$$d(Sx,Ty) \precsim \gamma(x,y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,Ty)+d(y,Sx)+d(y,x)}, \tag{2}$$

Then S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X and define the sequence $\{x_n\}$ be defined by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$,$n = 0,1,2,....$Now by (2), Then

$$\begin{aligned}
d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1})\\
&\precsim \gamma(x_{2n},x_{2n+1})\frac{d(x_{2n},Sx_{2n})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n})+d(x_{2n+1},x_{2n})}\\
&\precsim \gamma(x_{2n},x_{2n+1})\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n},x_{2n+2})+d(x_{2n+1},x_{2n+1})+d(x_{2n+1},x_{2n})}\\
&\precsim \gamma(x_{2n},x_{2n+1})\frac{d(x_{2n},x_{2n+1})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+1},x_{2n+2})}\\
&\precsim \gamma(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1}),
\end{aligned}$$

Taking the modulus, we get

$$|d(x_{2n+1},x_{2n+2})| \leq \gamma(x_{2n},x_{2n+1})|d(x_{2n},x_{2n+1})|.$$

Now by Proposition 2.1, therefore

$$\begin{aligned}
|d(x_{2n+1},x_{2n+2})| &\leq \gamma(x_0,x_{2n+1})|d(x_{2n},x_{2n+1})|\\
&\leq \gamma(x_0,x_1)|d(x_{2n},x_{2n+1})|
\end{aligned}$$

which yeilds that

$$|d(x_{2n+1},x_{2n+2})| \leq \gamma(x_0,x_1)|d(x_{2n},x_{2n+1})|.$$

Similarly, we get

$$\begin{aligned}
d(x_{2n+2},x_{2n+3}) &= d(Tx_{2n+1},Sx_{2n+2})\\
&\precsim \gamma(x_{2n+2},x_{2n+1})\frac{d(x_{2n+2},Sx_{2n+2})d(x_{2n+1},Tx_{2n+1})}{1+d(x_{2n+2},Tx_{2n+1})+d(x_{2n+1},Sx_{2n+2})+d(x_{2n+1},x_{2n+2})}\\
&\precsim \gamma(x_{2n+2},x_{2n+1})\frac{d(x_{2n+2},x_{2n+3})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+2},x_{2n+2})+d(x_{2n+1},x_{2n+3})+d(x_{2n+1},x_{2n+2})}\\
&\precsim \gamma(x_{2n+2},x_{2n+1})\frac{d(x_{2n+2},x_{2n+3})d(x_{2n+1},x_{2n+2})}{1+d(x_{2n+2},x_{2n+3})}\\
&\precsim \gamma(x_{2n+2},x_{2n+1})d(x_{2n+2},x_{2n+1}),
\end{aligned}$$

Taking the modulus, we get

$$|d(x_{2n+2},x_{2n+3})| \leq \gamma(x_{2n+2},x_{2n+1})|d(x_{2n+2},x_{2n+1})|.$$

Now by Proposition 2.1, therefore

$$\begin{aligned}
|d(x_{2n+2},x_{2n+3})| &\leq \gamma(x_0,x_{2n+1})|d(x_{2n+2},x_{2n+1})|\\
&\leq \gamma(x_0,x_1)|d(x_{2n+2},x_{2n+1})|
\end{aligned}$$

which yeilds that

$$|d(x_{2n+2},x_{2n+3})| \leq \gamma(x_0,x_1)|d(x_{2n+1},x_{2n+2})|.$$

Since $a = \gamma(x_0, x_1) < 1$,
thus we have,

$$|d(x_{2n+2}, x_{2n+3})| \leq a|d(x_{2n+1}, x_{2n+2})|,$$

or in fact

$$|d(x_n, x_{n+1})| \leq a|d(x_{n-1}, x_n)| \text{ for all } n \in N.$$

From lemma 1.6,we have $\{x_n\}$ is a Cauchy sequence in (X, d).Since X is complete,there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Next we show that u is a fixed point of S.
Now by (2) and Proposition 2.1, we can write

$$\begin{aligned} d(u, Su) &\precsim d(u, Tx_{2n+1}) + d(Tx_{2n+1}, Su) \\ &= d(u, Tx_{2n+1}) + d(Su, Tx_{2n+1}) \\ &\precsim d(u, Tx_{2n+1}) + \gamma(u, x_{2n+1}) \frac{d(u, Su)d(x_{2n+1}, Tx_{2n+1})}{1 + d(u, Tx_{2n+1}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)} \\ &\precsim d(u, x_{2n+2}) + \gamma(u, x_1) \frac{d(u, Su)d(x_{2n+1}, x_{2n+2})}{1 + d(u, x_{2n+2}) + d(x_{2n+1}, Su) + d(x_{2n+1}, u)}. \end{aligned}$$

on Making $n \to \infty$,reduces by making the modulus, we get

$$\begin{aligned} |d(u, Su)| &\leq \mu(u, x_1)|d(u, Su)| \\ &\leq (\gamma(u, x_1))|d(u, Su)| \\ &< |d(u, Su)|, \end{aligned}$$

which is contradiction. So, $Su = u$. Similarly, One can prove that u is a fixed point of T.by (2) and Proposition 2.1, we can write

$$\begin{aligned} d(u, Tu) &\precsim d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \\ &= d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\ &\precsim d(u, x_{2n+1}) + \gamma(x_{2n}, u) \frac{d(x_{2n}, Sx_{2n})d(u, Tu)}{1 + d(x_{2n}, Tu) + d(u, Sx_{2n}) + d(x_{2n}, u)} \\ &\precsim d(u, x_{2n+1}) + \gamma(x_0, u) \frac{d(x_{2n}, x_{2n+1})d(u, Tu)}{1 + d(x_{2n}, Tu) + d(u, x_{2n+1}) + d(x_{2n}, u)}. \end{aligned}$$

on Making $n \to \infty$,reduces to

$$d(u, Tu) \precsim \mu(x_0, u)d(u, Tu),$$

Taking the modulus, we get

$$\begin{aligned} |d(u, Tu)| &\leq \mu(x_0, u)|d(u, Tu)| \\ &\leq (\gamma(x_0, u))|d(u, Tu)| \\ &< |d(u, Tu)|, \end{aligned}$$

which is contradiction. So, $Tu = u$. We present to prove the uniqueness of the common fixed point of S and T. For this,Assume that the existence u^* is a second common fixed point. we have

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\precsim \gamma(u, u^*) \frac{d(u, Su)d(u^*, Tu^*)}{1 + d(u, Tu^*) + d(u^*, Su) + d(u, u^*)} \end{aligned}$$

which implies that

$$d(u, u^*) = 0.$$

Thus $u = u^*$, completing the proof of the theorem. □

Corollary 2.3. *Let (X, d) be a complex-valued metric space and let $S : X \to X$. If there exists control function $\gamma : X \times X \to [0, 1)$ such that for all $x, y \in X$ we have*

$$\gamma(S^2x, y) \leq \gamma(x, y) \text{ and } \gamma(x, S^2y) \leq \gamma(x, y);$$

$$\gamma(x, y) < 1;$$

$$d(Sx, Sy) \preceq \gamma(x, y)\frac{d(x, Sy)d(y, Sx)}{1 + d(x, Sy) + d(y, Sx) + d(x, y)};$$

then S have a unique fixed point.

Proof. Take $T = S$ in Theorem 2.2 □

Corollary 2.4. *Let (X, d) be a complex valued metric space and let $S, T : X \to X$. If there exists constants $\gamma > 0$ such that*

$$\gamma < 1;$$

and for all $x, y \in X$ we have

$$d(Sx, Ty) \preceq \gamma\frac{d(x, Sy)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(x, y)};$$

then S and T have a unique common fixed point.

Proof. Take γ a constant functions in Theorem 2.2. □

Example 2.5. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{C}$

$$d(x, y) = |x - y| + i|x - y|$$

for all $x, y \in X$. Then (X, d) is a complex metric space. Now we define the mappings $S, T : X \to X$ by

$$S(x) = \frac{x}{6} \text{ and } T(y) = \frac{y}{6}.$$

Consider the functions $\gamma : X \times X \to [0, 1)$

$$\gamma(x, y) = \frac{x^2y^2}{30}.$$

Clearly $\gamma(x_0, x_1) < 1$.
We satisfy the condition (a)of main theorem 2.2 as follows.

$$\begin{aligned}\gamma(TSx, y) =& \gamma(T(\frac{x}{6}), y) = \gamma(\frac{x}{36}, y)\\ &\leq \gamma(x, y),\end{aligned}$$

That is $\gamma(TSx, y) \leq \gamma(x, y)$, for all $x, y \in X$.
And

$$\begin{aligned}\gamma(x, STy) =& \gamma(x, S(\frac{y}{6})) = \gamma(x, \frac{y}{36})\\ &\leq \gamma(x, y),\end{aligned}$$

That is $\gamma(x, STy) \leq \gamma(x, y)$, for all $x, y \in X$.

Now for the verification of condition (c), we have for all $x, y \in X$

$$0 \precsim \frac{d(x, Sx)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(y, x)}.$$

Consider

$$\begin{aligned}d(Sx, Ty) =& d(\frac{x}{6}, \frac{y}{6}) = |\frac{x}{6} - \frac{y}{6}| + i|\frac{x}{6} - \frac{y}{6}|\\ =& \frac{1}{6}(|x - y| + i|x - y|)\\ \precsim \gamma(x, y)&\frac{d(x, Sx)d(y, Ty)}{1 + d(x, Ty) + d(y, Sx) + d(y, x)}\end{aligned}$$

Therefore all the conditions of Theorem 2.2 are satisfied and $x = 0 \in X$ is a unique common fixed point of S and T.

3. Conclusion

This paper has explored the dynamic realm of fixed point theory, particularly within the intricate domain of contraction mappings in complex metric spaces. By introducing the concept of control functions, we have shed new light on the behavior and properties of fixed points, enriching our understanding of their convergence properties. Our investigation highlights the symbiotic relationship between contraction mappings and complex metric spaces, underscoring the indispensable role of control functions in shaping the trajectory of fixed-point iterations. Through our analysis, we have not only advanced the theoretical framework of fixed point theory but also opened avenues for further exploration and application in various mathematical and scientific domains.

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Declarations

The author declares that he has no conflicts of interest.

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