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Error estimation for a piezoelectric contact problem with wear and long memory

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abstract

We study a mathematical model for a quasistatic behavior of electro-viscoelastic materials. The problem is related to highly nonlinear and non-smooth phenomena like contact, friction and normal compliance with wear. Then, a fully discrete scheme is introduced based on the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. For a numerical scheme, we prove the existence and uniqueness of the solutions, and derive optimal order error estimates under certain regularity assumption on the solution of the continuous problem.

keywords

Quasistatic process, electro-viscoelastic materials, friction, wear, fully discrete scheme, error estimates.

2020 Mathematics Subject Classification

35J85 · 49J40 · 47J20 · 74M15

1. Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials. It consists of the appearance of electric charges on the surfaces of some crystals after their deformation. Conversely, experiments have shown that the action of an electric field on the crystals can generate stresses and deformations. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in Batra and Yang, 1995 and Ikeda, 1996. In Moumen and Rebiai, 2024 , the authors examine a transmission system of the Schrödinger equation with Neumann feedback control, which includes a time-varying delay term and acts on the exterior boundary. They utilize an appropriate energy function and a suitable Lyapunov functional. The authors of Acil et al., 2024 demonstrate the system's robustness, stability, and ability to respond to fast changes, making it a promising solution for efficient energy management in hybrid PV-battery systems. A static frictional contact problem for electric-elastic materials was considered in Maceri and Bisegna, 1998 and Migórski, 2006. Contact problems with friction or adhesion for electro-viscoelastic materials were studied in Selmani and Selmani, 2010 and Lerguet et al., 2007 and recently in Migórski et al., 2011 in the case of an electrically conductive foundation.

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In this paper we consider a mathematical model for the process of contact with normal compliance and friction contact conditions when the wear of the contact surface due to friction is taken into account. The foundation is assumed to move steadily and only sliding contact takes places. The material is electro-viscoelastic with long memory, defined by a relaxation operator.

This work constitutes in some sense a continuation paper of the results obtained in Selmani, 2013. The work in Selmani, 2013 has been devoted to a qualitative results like existence and uniqueness result of weak solutions on displacement, electric potential and wear fields have been proved but no numerical approximations have been performed. Here we follow the latter work and propose a numerical scheme for the approximation of the solution fields so as to elaborate a general numerical analysis of error estimates.

The main goal of this work is to formulate an approximate solution of our problem, which can quickly converge to the exact solution. For that, this work is organized as follows. In Section 3 we give a short description of the mathematical model and recall the main existence and uniqueness result. In Section 4, For the numerical scheme, we prove the existence and uniqueness of the solutions. Finally, in Section 5, we derive optimal-order error estimates under certain regularity assumptions on the solution of the continuous problem.

2. Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\ .\ |$ will represent the inner product and the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces:

$$H = \left\{ \boldsymbol{u} = (u_i)\ /\ u_i \in L^2(\Omega) \right\},$$

$$\mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij})\ /\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\},$$

$$H_1 = \{ \boldsymbol{u} = (u_i)\ /\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H} \},$$

$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H}\ /\ Div\boldsymbol{\sigma} \in H \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ Div\boldsymbol{\sigma} = (\sigma_{ij,\,j}).$$

A subscript that follows a comma indicates a partial derivative with respect to the corresponding spatial variable, e.g., $u_{i,j} = \partial u_i / \partial x_j$.

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i\ dx,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij}\ dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H\ .$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|.|_H$, $|.|_{\mathcal{H}}$, $|.|_{H_1}$ and $|.|_{\mathcal{H}_1}$, respectively. For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}.\boldsymbol{\nu},\ \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu},\ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{v}.\boldsymbol{v} da\ \ \forall\, \boldsymbol{v} \in H_1. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0, T; X)$ and $C^1(0, T; X)$ for the space of continuous and continuously differentiable functions from $[0, T]$ to X, respectively. We use dots for derivatives with respect to the time variable t.

The space $C(0, T; X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X$$

while $C^1(0, T; X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X + \max_{t\in[0,T]} \left|\dot{\boldsymbol{f}}(t)\right|_X .$$

Finally, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue spaces $L^p(0, T; X)$ and for the Sobolev spaces $W^{k,p}(0, T; X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Statement of the problem

An electro-viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body is subjected to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. We consider a partition of Γ into three disjoint measurable subsets Γ_1, Γ_2 and Γ_3, on one hand, and on two disjoint measurable subsets Γ_a and Γ_b, on the other hand, such that *meas* $(\Gamma_1) > 0$, *meas* $(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on Γ_1, so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface free electrical charge of density q_2 is prescribed on Γ_b. In the reference configuration, the body may come in contact over Γ_3 with a conductive obstacle, which is also called the foundation. The contact is frictional and is modeled with normal compliance, taking into account the wear of the contact surfaces. The foundation is assumed to move steadily and only sliding contact takes places. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected.

We are interested in the evolution of the deformation of the body and of the electric potential on the time interval $[0, T]$. The process is assumed to be isothermal, electrically static, i.e., all radiation effects are neglected, and mechanically quasistatic, i.e., the inertial terms in the momentum balance equations are neglected. To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\boldsymbol{x} \in \Omega\cup\Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of sliding frictional contact problem with normal compliance and wear may be stated as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, *an electric potential field* $\varphi : \Omega \times [0, T] \to \mathbb{R}$, *an electric displacement field* $\boldsymbol{D} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a* wear function $\zeta : \Gamma_3 \times [0, T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds \tag{3.1}$$

$$+\mathcal{E}^*\nabla\varphi(t) \text{ in } \Omega \times (0, T),$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{B}\nabla\varphi \text{ in } \Omega \times (0, T), \tag{3.2}$$

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \text{ in } \Omega \times (0, T), \tag{3.3}$$

$$div\boldsymbol{D} = q_0 \text{ in } \Omega \times (0, T), \tag{3.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_1 \times (0, T), \tag{3.5}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \text{ on } \Gamma_2 \times (0, T), \tag{3.6}$$

$$\begin{cases} -\sigma_\nu = p_\nu (u_\nu - g - \zeta), \\ |\boldsymbol{\sigma}_\tau| = p_\tau (u_\nu - g - \zeta), \\ \boldsymbol{\sigma}_\tau = -\lambda (\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*), \ \lambda \geq 0, \\ \dot{\zeta} = -k_0 v^* \sigma_\nu, \end{cases} \text{ on } \Gamma_3 \times (0, T), \tag{3.7}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0, T), \tag{3.8}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0, T), \tag{3.9}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \zeta(0) = 0 \text{ in } \Omega. \tag{3.10}$$

Here, equations (3.1) – (3.2) represent the constitutive law for a piezoelectric material with long memory where $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathcal{M}$ is a relaxation fourth order tensor. $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$is its transposed and $\boldsymbol{B}$ denotes the electric permittivity tensor. Equations (3.3) and (3.4) represent the equilibrium equations for the stress and electric-displacement fields. Equations (3.5) and (3.6) are the displacement-traction boundary conditions, respectively. (3.7) represents the condition with normal compliance, friction and wear where g represents the initial gap between the body and the foundation, $k_0 > 0$ is a wear coefficient and $\boldsymbol{v}^*$ is the tangential velocity of the foundation such that $v^* = |\boldsymbol{v}^*|$. Equations (3.8) and (3.9) represent the electric boundary conditions. In (3.10) $\boldsymbol{u}_0$ is the given initial displacement and $\zeta(0) = 0$ means that at the initial moment the body is not subject to any prior wear.

To obtain a variational formulation of the problem (3.1) – (3.10) we introduce the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1 \ / \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $c_k > 0$ which depends only on Ω and Γ_1 such that

$$|\boldsymbol{\varepsilon}(\boldsymbol{v}|_{\mathcal{H}} \geq c_k \, |\boldsymbol{v}|_{H_1} \, \forall \boldsymbol{v} \in V.$$

On the space V we consider the inner product and the associated norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(v))_{\mathcal{H}} \, , \ \ |\boldsymbol{v}|_V = |\boldsymbol{\varepsilon}(\boldsymbol{v}|_{\mathcal{H}} \ \ \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.11}$$

It follows from Korn's inequality that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V. Therefore $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and (3.11), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \leq c_0 \, |\boldsymbol{v}|_V \ \ \forall \boldsymbol{v} \in V. \tag{3.12}$$

We also introduce the spaces.

$$W = \left\{ \phi \in H^1(\Omega) \ / \ \phi = 0 \text{ on } \Gamma_a \right\},$$

$$\mathcal{W} = \left\{ \boldsymbol{D} = (D_i) \ / \ D_i \in L^2(\Omega), \ div\boldsymbol{D} \in L^2(\Omega) \right\},$$

where $div\boldsymbol{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi . \nabla\phi \; dx, \tag{3.13}$$

$$(\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = \int_\Omega \boldsymbol{D}.\boldsymbol{E} \; dx + \int_\Omega div\boldsymbol{D}.div\boldsymbol{E} \; dx. \tag{3.14}$$

The associated norms will be denoted by $|.|_W$ and $|.|_{\mathcal{W}}$, respectively. Moreover, when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{D}, \nabla\phi)_H + (div\boldsymbol{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}.\boldsymbol{\nu} \; \phi da \ \ \forall \phi \in H^1(\Omega).$$

Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$|\nabla\phi|_H \geq c_F \, |\phi|_{H^1(\Omega)} \, \forall \phi \in W, \tag{3.15}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. It fallows from (3.15) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therfore $(W, |.|_W)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem and (3.13), there exists a constant $a_0 > 0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\phi|_{L^2(\Gamma_3)} \leq a_0 \, |\phi|_W \ \ \forall \, \phi \in W. \tag{3.16}$$

In the study of the mechanical problem (3.1) – (3.10), we make the following assumptions. Assume that the operators $\mathcal{A}$, $\mathcal{F}$, $\mathcal{E}$, $\boldsymbol{B}$ and the functions p_r $(r = \nu, \tau)$ satisfy the following conditions with $L_{\mathcal{A}}$, $m_{\mathcal{A}}$, $L_{\mathcal{F}}$, L_r and m_r being positive constants:

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \\ \text{(b) } |\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2 \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue} \\ \quad \text{measurable in } \Omega \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{3.17}$$

$$\begin{cases} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d \\ \text{(b) } |\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{F}} |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) The mapping } \boldsymbol{x} \to \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) } \boldsymbol{x} \to \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{3.18}$$

$$\begin{cases} \text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d \\ \text{(b) } \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau} = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \\ \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.19}$$

$$\begin{cases} \text{(a) } \boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \\ \text{(b) } \boldsymbol{B}(\boldsymbol{x})\boldsymbol{E} = (b_{ij}(x)E_j) \\ \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) } b_{ij} = b_{ji}, \ b_{ij} \in L^\infty(\Omega). \\ \text{(d) } \boldsymbol{B}\boldsymbol{E}.\boldsymbol{E} \geq m_B |\boldsymbol{E}|^2 \\ \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.20}$$

$$\begin{cases} \text{(a) } p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \ (r = \nu, \tau) \\ \text{(b) } |p_r(\boldsymbol{x}, \alpha_1) - p_r(\boldsymbol{x}, \alpha_2)| \leq L_r |\alpha_1 - \alpha_2| \\ \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) } |p_r(\boldsymbol{x}, \alpha)| \leq m_r \ \forall \alpha \in \mathbb{R}, \text{ p.p. } \boldsymbol{x} \in \Gamma_3, \\ \text{(d) The mapping } \boldsymbol{x} \to p_r(\boldsymbol{x}, \alpha) \text{ is Lebesgue} \\ \quad \text{measurable on } \Gamma_3 \text{ for any } \alpha \in \mathbb{R}. \\ \text{(e) } \boldsymbol{x} \to p_r(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.21}$$

The relaxation tensor $\mathcal{M}$ satisfies

$$\mathcal{M} \in C(0, T; \mathcal{H}_\infty), \tag{3.22}$$

where $\mathcal{H}_\infty$ is the space of fourth order tensor field given by

$$\mathcal{H}_\infty = \{\boldsymbol{E} = (E_{ijkl}) \ / \ E_{ijkl} = E_{jikl} = E_{klij} \in L^\infty(\Omega), \ 1 \leq i, j, k, l \leq d\},$$

which is a real Banach space with the norm

$$|\boldsymbol{E}|_{\mathcal{H}_\infty} = \sum_{1 \leq i,j,k,l \leq d} |E_{ijkl}|_{L^\infty(\Omega)}.$$

The density of volume forces, traction, volume electric charges and surface electric charges have the regularity

$$\boldsymbol{f}_0 \in C(0, T; H), \ \boldsymbol{f}_2 \in C(0, T; L^2(\Gamma_2)^d), \tag{3.23}$$

$$q_0 \in C(0, T; L^2(\Omega)), \ q_2 \in C(0, T; L^2(\Gamma_b)). \tag{3.24}$$

$$q_2 = 0 \text{ on } \Gamma_3 \ \forall t \in [0, T]. \tag{3.25}$$

We assume that the gap function g and the initial displacement field $\boldsymbol{u}_0$ satisfy

$$g \in L^2(\Gamma_3), \ g \geq 0 \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{3.26}$$

$$\boldsymbol{u}_0 \in V. \tag{3.27}$$

We define the three mappings $\boldsymbol{f} : [0, T] \to V$, $q : [0, T] \to W$ and $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$, respectively, by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t).\boldsymbol{v} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t).\boldsymbol{v} da, \tag{3.28}$$

$$(q(t), \phi)_W = \int_\Omega q_0(t)\phi dx - \int_{\Gamma_b} q_2(t)\phi da. \tag{3.29}$$

$$j(\boldsymbol{u}, \boldsymbol{v}, \zeta) = \int_{\Gamma_3} p_\nu(u_\nu - g - \zeta) v_\nu da \tag{3.30}$$

$$+ \int_{\Gamma_3} p_\tau(u_\nu - g - \zeta) |\boldsymbol{v}_\tau - \boldsymbol{v}^*| da,$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\zeta \in L^2(\Gamma_3)$ and $t \in [0,T]$. The functional $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} \text{For all } \boldsymbol{u} \in V \text{ and } \zeta \in L^2(\Gamma_3),\ \boldsymbol{v} \to j(\boldsymbol{u}, \boldsymbol{v}, \zeta) \\ \text{is proper, convex and lower semicontinuous on } V. \end{array} \right.$$

We note that condition (3.23) and (3.24) imply that

$$\boldsymbol{f} \in C(0,T;V),\ q \in C(0,T;W). \tag{3.31}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.1) – (3.10).

Problem VP. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$, *an electric potential field* $\varphi : [0,T] \to W$, *an electric displacement field* $\boldsymbol{D} : [0,T] \to \mathcal{W}$ *and a wear function* $\zeta : [0,T] \to L^2(\Gamma_3)$ *such that for all* $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{M}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds + \mathcal{E}^*\nabla\varphi(t), \tag{3.32}$$

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{v}, \zeta(t)) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \zeta(t)) \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{3.33}$$

$$\boldsymbol{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{B}\nabla\varphi(t), \tag{3.34}$$

$$(\boldsymbol{D}(t), \nabla\phi)_H = -(q(t), \phi)_W \quad \forall \phi \in W, \tag{3.35}$$

$$\dot{\zeta} = k_0 v^* p_\nu (u_\nu - g - \zeta), \tag{3.36}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \zeta(0) = 0, \tag{3.37}$$

The main result in this section is the following existence and uniqueness result (see for details Selmani, 2013).

Theorem 3.1. *Assume that* (3.17) – (3.27) *hold. Then, there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \zeta\}$ *to Problem VP. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{3.38}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{3.39}$$

$$\varphi \in C(0,T;W), \tag{3.40}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}), \tag{3.41}$$

$$\zeta \in C^1(0,T;L^2(\Gamma_3)). \tag{3.42}$$

4. Fully discrete approximation

In this section, we introduce a discrete numerical scheme of Problem VP. We assume that the conditions (3.17) – (3.27) hold. Thus, it follows from Theorem 3.1 that Problem VP has a unique solution. More precisely, we are interested in solving Problem VP over a finite time interval $[0,T]$, with $T > 0$ arbitrary but fixed. Thus, let N be a positive integer; we define the time step size $k = \frac{T}{N}$ and we consider the uniform time discretization $t_n = nk$, $0 \leq n \leq N$, where N is a sufficiently large integer. For a continuous function $v(t)$ with values in a function space, we write $v_j = v(t_j)$, $0 \leq j \leq N$. For spatial discretization, we consider a polygonal domain Ω.

For the discretization of the integrals, we use the rectangle method

$$\int_{t_j}^{t_{j+1}} v(s)ds = kv_j.$$

Let $\mathcal{H}^h$ and B^h be the finite element spaces of piecewise constants. The spaces $\mathcal{H}$ and $L^2(\Gamma_3)$ are approximated by $\mathcal{H}^h$ and B^h, respectively.

The V and W spaces are approximated respectively by the following finite element spaces:

$$V^h = \left\{ \boldsymbol{v}^h \in \left[C\left(\overline{\Omega}\right)\right]^d \ \mid\ \boldsymbol{v}^h|_K \in [P_1(K)]^d\ \forall K \in \mathcal{T}_h,\ \boldsymbol{v}^h = 0 \text{ on } \Gamma_1 \right\},$$

$$W^h = \left\{ \phi^h \in C\left(\overline{\Omega}\right) \ \mid\ \phi^h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h,\ \phi^h = 0 \text{ on } \Gamma_a \right\},$$

where $\mathcal{T}_h$ is an element derived from the triangularization of $\overline{\Omega}$, $P_1(K)$ is the space of polynomials of degree smaller or equal to one on K and h refers to the spatial discretion parameter which is defined as

$$h = \max_{K \in \mathcal{T}_h} \operatorname{diam}(K), \text{with } \operatorname{diam}(K) = \max\{|x-y|;\ x, y \in K\}.$$

For all $\boldsymbol{\tau} \in \mathcal{H}$, $\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}$ is the orthogonal projection of finite elements on $\mathcal{H}^h$,

$$\left(\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} = \left(\boldsymbol{\tau}, \boldsymbol{\tau}^h\right)_{\mathcal{H}} \ \forall \boldsymbol{\tau}^h \in \mathcal{H}^h.$$

It is convenient to introduce the velocity field

$$\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t) \text{ so } \boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}(s)ds, t \in [0, T].$$

It follows from Theorem 3.1 that $\boldsymbol{v} \in C(0, T; V)$ and for all $t \in [0, T]$, we have

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{M}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds + \mathcal{E}^*\nabla\varphi(t), \tag{4.1}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{v}, \zeta(t)) - j(\boldsymbol{u}(t), \boldsymbol{v}(t), \zeta(t)) \\ &\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}(t))_V \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{4.2}$$

Let $\boldsymbol{u}_0^h \in V^h$ be a finite element approximation of $\boldsymbol{u}_0$.

The fully discrete approximation of Problem VP is the following.

Problem VP^{hk}. *Find a discrete velocity field* $\boldsymbol{v}^{hk} = \left\{\boldsymbol{v}_n^{hk}\right\}_{n=0}^N \subset V^h$, *a discrete stress field* $\boldsymbol{\sigma}^{hk} = \left\{\boldsymbol{\sigma}_n^{hk}\right\}_{n=0}^N \subset \mathcal{H}^h$, *a discrete electric potential* $\varphi^{hk} = \left\{\varphi_n^{hk}\right\}_{n=0}^N \subset W^h$ *and a discrete wear field* $\zeta^{hk} = \left\{\zeta_n^{hk}\right\}_{n=0}^N \subset B^h$ *such that*

$$\boldsymbol{\sigma}_0^h = \mathcal{P}_{\mathcal{H}^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h) + \mathcal{P}_{\mathcal{H}^h}\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h) + \mathcal{P}_{\mathcal{H}^h}\mathcal{E}^*\nabla\varphi_0^h, \tag{4.3}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_0^h, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_0^h))_{\mathcal{H}} + j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0) \\ &\geq (\boldsymbol{f}(0), \boldsymbol{v}^h - \boldsymbol{v}_0^h)_V \quad \forall \boldsymbol{v}^h \in V^h, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &(\boldsymbol{B}\nabla\varphi_0^h, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\phi^h)_H \\ &= (q(0), \phi^h)_W \quad \forall \phi^h \in W^h, \end{aligned} \tag{4.5}$$

and for $n \geq 1$,

$$\begin{aligned} \boldsymbol{\sigma}_n^{hk} = {}& \mathcal{P}_{\mathcal{H}^h}\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + \mathcal{P}_{\mathcal{H}^h}\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \mathcal{P}_{\mathcal{H}^h}\mathcal{E}^*\nabla\varphi_n^{hk} \\ &+ k\sum_{j=0}^{n-1}\mathcal{P}_{\mathcal{H}^h}\left(\mathcal{R}_n\right)_j^{hk} \end{aligned} \tag{4.6}$$

$$\begin{aligned} &(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk}) \\ &\geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{v}_n^{hk})_V \quad \forall \boldsymbol{v}^h \in V^h, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &(\boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\phi^h)_H \\ &= (q_n, \phi^h)_W \quad \forall \phi^h \in W^h, \end{aligned} \tag{4.8}$$

$$\zeta_n^{hk} = kk_0\upsilon^*\sum_{j=0}^{n-1} p_\nu\left(u_{\nu j}^{hk} - g - \zeta_j^{hk}\right). \tag{4.9}$$

Here, we used the following notations

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h,\ \boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h,\ \boldsymbol{\sigma}_0^{hk} = \boldsymbol{\sigma}_0^h,\ \varphi_0^{hk} = \varphi_0^h \text{ and } \zeta_0^{hk} = \zeta_0^h = \zeta_0 = 0.$$

We use the following discrete displacement field

$$\boldsymbol{u}_n^{hk} = \boldsymbol{u}_0^h + k\sum_{j=1}^{n}\boldsymbol{v}_j^{hk} \ n \geq 1, \tag{4.10}$$

We also use the notations

$$\begin{cases} \left(\mathcal{R}_n\right)_j^{hk} = \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}), \\ \left(\mathcal{R}_n\right)(s) = \mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \\ \left(\mathcal{R}_n\right)_j = \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j). \end{cases} \tag{4.11}$$

We have the following existence and uniqueness result.

Theorem 4.1. *Suppose that the conditions stated in Theorem 3.1 are satisfied. Then the Problem VP^{hk} has a unique solution.*

Proof. First,we show that (4.3) – (4.5) uniquely determines $\sigma_0^h \in \mathcal{H}^h$, $\upsilon_0^h \in V^h$ and $\varphi_0^h \in W^h$. From a discrete analogue of Lemma 4.5 in Selmani, 2013, it follows that (4.5) has a unique solution $\varphi_0^h \in W^h$. Combining (4.3) and (4.4), we obtain an elliptic variational inequality which has a unique solution $\boldsymbol{v}_0^h \in V^h$. $\boldsymbol{\sigma}_0^h \in \mathcal{H}^h$ is then calculated from (4.3).

Next, we show that with $\left\{\left(\boldsymbol{\sigma}_j^{hk}, \boldsymbol{v}_j^{hk}, \varphi_j^{hk}, \zeta_j^{hk}\right)\right\}_{j\leq n-1} \subset \mathcal{H}^h \times V^h \times W^h \times B^h$ known, (4.6) – (4.9) uniquely determines $\left(\boldsymbol{\sigma}_n^{hk}, \boldsymbol{v}_n^{hk}, \varphi_n^{hk}, \zeta_n^{hk}\right) \subset \mathcal{H}^h \times V^h \times W^h \times B^h$. Given $\left\{\left(\boldsymbol{u}_j^{hk}, \zeta_j^{hk}\right)\right\}_{j\leq n-1} \in V^h \times B^h$, a discrete analogue of Lemma 4.5 in Selmani, 2013 shows that (4.8) has a unique solution $\varphi_n^{hk} \in W^h$ and $\zeta_n^{hk} \in B^h$ is computed from (4.9).

Finally, combining (4.6) and (4.7), we obtain

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \boldsymbol{v}_n^{hk}))_{\mathcal{H}} + j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}^h, \zeta_n^{hk}) - j(\boldsymbol{u}_{n-1}^{hk}, \boldsymbol{v}_n^{hk}, \zeta_n^{hk}) \tag{4.12}$$

$$\geq (\boldsymbol{r}_n, \boldsymbol{v}^h - \boldsymbol{v}_n^{hk})_V \quad \forall \boldsymbol{v}^h \in V^h,$$

where

$$(\boldsymbol{r}_n, \boldsymbol{v}^h)_V = (\boldsymbol{f}_n, \boldsymbol{v}^h)_V - \Big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + \mathcal{E}^*\nabla\varphi_n^{hk} \tag{4.13}$$

$$\left. + k\sum_{j=0}^{n-1} (\mathcal{R}_n)_j^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}^h)\right)_{\mathcal{H}}.$$

By a standard result on elliptic variational inequalities, there exists a unique $\boldsymbol{v}_n^{hk} \in V^h$ satisfying (4.12). We compute $\boldsymbol{\sigma}_n^{hk}$ from (4.6). □

5. Error estimates

This section is devoted to deriving error estimates for the discrete solution. We make the following solution regularity assumptions:

$$(\mathcal{M}, \boldsymbol{u}, \zeta) \in C^1(0, T; \mathcal{H}_\infty \times V \times L^2(\Gamma_3)), \tag{5.1}$$

$$(\boldsymbol{v}, \boldsymbol{\sigma}, \varphi) \in C(0, T; V \times \mathcal{H}_1 \times W), \tag{5.2}$$

$$(\boldsymbol{v}, \boldsymbol{\sigma}, \varphi) \in C(0, T; H^2(\Omega)^d \times H^1(\Omega)^{d\times d} \times H^2(\Omega)), \tag{5.3}$$

$$\boldsymbol{v} \in C\left(0, T; H^2(\Gamma_3)^d\right), \boldsymbol{\sigma\nu} \in C(0, T; L^2(\Gamma)^d), \boldsymbol{u}_0 \in H^2(\Omega)^d. \tag{5.4}$$

In this section, no summation is assumed over a repeated index and c denotes a positive constant which depends on the problem data, but is independent on the discretization parameters, h and k.

Lemma 5.1. *Assume that* (3.17) – (3.27) *hold. Let* $\{\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{u}, \varphi, \zeta\}$ *and* $\left\{\boldsymbol{\sigma}_n^{hk}, \boldsymbol{v}_n^{hk}, \boldsymbol{u}_n^{hk}, \varphi_n^{hk}, \zeta_n^{hk}\right\}$ *denote the solution to Problems* **VP** *and* **VP**^{hk}*, respectively. Then, the following error estimates hold for all* $\boldsymbol{v}^h \in V^h$ *and* $\phi^h \in W^h$ *:*

$$\max_{0\leq n\leq N}\left\{\begin{array}{c} \left|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}} + \left|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right|_V + \left|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right|_V \\ + \left|\varphi_n - \varphi_n^{hk}\right|_W + \left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} \end{array}\right\} \tag{5.5}$$

$$\leq ck + c\left\{\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V + \max_{0\leq n\leq N} \left|(I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}} + \max_{0\leq n\leq N}\inf_{\phi^h\in W^h}\left|\varphi_n - \phi^h\right|_W\right.$$

$$\left. + \max_{0\leq n\leq N}\inf_{\boldsymbol{v}^h\in V^h}\left(\left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_V + \left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d}^{\frac{1}{2}}\right)\right\}.$$

Proof. First, we make an error estimate on the electric potential. We combine (3.34) and (3.35), we have for all $t \in [0, T]$ and $\phi \in W$,

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\phi)_H = (q(t), \phi)_W. \tag{5.6}$$

Taking (5.6) at $t = t_n$ and for all $\phi = \phi^h \in W^h$ and $n \geq 1$, it follows that

$$(\boldsymbol{B}\nabla\varphi_n, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n), \nabla\phi^h)_H = (q(t), \phi^h)_W. \tag{5.7}$$

We subtract (4.8) from (5.7) to obtain for all $\phi^h \in W^h$ and $n \geq 1$

$$(\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\phi^h)_H = 0,$$

thus

$$(\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\phi^h - \varphi_n^{hk}\right))_H = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\phi^h - \varphi_n^{hk}\right))_H,$$

using the writing $\phi^h = \phi^h + \varphi_n - \varphi_n$, we see that

$$\begin{aligned} & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ = & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \phi^h\right))_H. \end{aligned}$$

Using (3.20) to see that

$$\begin{aligned} m_B \left|\varphi_n - \varphi_n^{hk}\right|_W^2 \leq & (\boldsymbol{B}\nabla\varphi_n - \boldsymbol{B}\nabla\varphi_n^{hk}, \nabla\left(\varphi_n - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \varphi_n^{hk}\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}), \nabla\left(\varphi_n - \phi^h\right))_H, \end{aligned}$$

using the Cauchy-Schwarz inequality and the following inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2 \ \forall \epsilon > 0, \tag{5.8}$$

we obtain

$$\left|\varphi_n - \varphi_n^{hk}\right|_W^2 \leq c\left(\left|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right|_V^2 + \left|\varphi_n - \phi^h\right|_W^2\right). \tag{5.9}$$

From (5.6) at $t = 0$ with $\phi = \phi^h \in W^h$, we have

$$(\boldsymbol{B}\nabla\varphi_0, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \nabla\phi^h)_H = (q\left(0\right), \phi^h)_W,$$

We subtract (4.5) from the previous equality to obtain

$$(\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\phi^h)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\phi^h)_H = 0,$$

then, we can write

$$(\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\phi^h - \varphi_0^h\right))_H = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\phi^h - \varphi_0^h\right))_H.$$

We use the writing $\phi^h = \phi^h - \varphi_0 + \varphi_0$ to note

$$\begin{aligned} & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ = & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \phi^h\right))_H. \end{aligned}$$

By using (3.20) to see that

$$\begin{aligned} m_B \left|\varphi_0 - \varphi_0^h\right|_W^2 \leq & (\boldsymbol{B}\nabla\varphi_0 - \boldsymbol{B}\nabla\varphi_0^h, \nabla\left(\varphi_0 - \phi^h\right))_H \\ & +(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \varphi_0^h\right))_H \\ & -(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h), \nabla\left(\varphi_0 - \phi^h\right))_H, \end{aligned}$$

Using the inequality of Cauchy-Schwarz, (3.19) – (3.20) and (5.8), we find

$$\left|\varphi_0 - \varphi_0^h\right|_W^2 \leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \phi^h\right|_W^2\right). \tag{5.10}$$

Next, we state two relations that we will use in error estimations (see Sofonea et al., 2005)

$$\left|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right|_V^2 \leq ck^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + ck\sum_{j=1}^{n}\left|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right|_V^2, \tag{5.11}$$

$$\left|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}^{hk}\right|_V^2 \leq ck^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + ck\sum_{j=0}^{n-1}\left|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right|_V^2. \tag{5.12}$$

We note for all $n \geq 1$

$$\begin{aligned}\theta_n^{hk}(\mathcal{R}_n) &= \int_0^{t_n} (\mathcal{R}_n)(s)ds - \sum_{j=0}^{n-1} k (\mathcal{R}_n)_j^{hk} \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[(\mathcal{R}_n)(s) - (\mathcal{R}_n)_j\right] ds \\ &\quad + \sum_{j=0}^{n-1} k \left[(\mathcal{R}_n)_j - (\mathcal{R}_n)_j^{hk}\right],\end{aligned}$$

then

$$\theta_n^{hk}(\mathcal{R}_n) = I_n + I_n^{hk}, \tag{5.13}$$

where

$$I_n = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[(\mathcal{R}_n)(s) - (\mathcal{R}_n)_j\right] ds,\ \ I_n^{hk} = \sum_{j=0}^{n-1} k \left[(\mathcal{R}_n)_j - (\mathcal{R}_n)_j^{hk}\right].$$

We have

$$\begin{aligned} I_n &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j)\right] ds \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}_j)\right] ds \\ &\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\mathcal{M}(t_n - s) - \mathcal{M}(t_n - t_j)\right] \boldsymbol{\varepsilon}(\boldsymbol{u}_j) ds.\end{aligned}$$

We use the hypothesis (3.22), we obtain

$$|I_n|_{\mathcal{H}} \leq c \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[|\boldsymbol{u}(s) - \boldsymbol{u}_j|_V + |\mathcal{M}(t_n - s) - \mathcal{M}(t_n - t_j)|_{\mathcal{H}_\infty}\right] ds.$$

Using (5.1), the sum can be bounded by ck where the constant c is proportional to $|\dot{\boldsymbol{u}}|_{C(0,T;V)} + \left|\dot{\mathcal{M}}\right|_{C(0,T;\mathcal{H}_\infty)}$. Hence

$$|I_n|_{\mathcal{H}}^2 \leq ck^2. \tag{5.14}$$

We also have

$$I_n^{hk} = \sum_{j=0}^{n-1} k \left[\mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j) - \mathcal{M}(t_n - t_j)\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk})\right],$$

From (3.22) and (3.11), we find

$$\left|I_n^{hk}\right|_{\mathcal{H}} \leq ck \sum_{j=0}^{n-1} \left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V. \tag{5.15}$$

We combine (5.13) – (5.14) and (5.15) to see that

$$\left|\theta_n^{hk}(\mathcal{R}_n)\right|_{\mathcal{H}}^2 \leq ck^2 + ck \sum_{j=0}^{n-1} \left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V^2. \tag{5.16}$$

Furthermore, we apply (4.1) at $t = t_n$ to see that

$$\boldsymbol{\sigma}_n = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + \int_0^{t_n} \mathcal{M}(t_n - s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds + \mathcal{E}^*\nabla\varphi_n. \tag{5.17}$$

Using (4.6) and (5.17), we can write for all $n \geq 1$

$$\begin{aligned}\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} &= (I - \mathcal{P}_{\mathcal{H}^h})\boldsymbol{\sigma}_n + \mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk} \\ &= (I - \mathcal{P}_{\mathcal{H}^h})\boldsymbol{\sigma}_n + \mathcal{P}_{\mathcal{H}^h}\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})\right) + \left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})\right)\right.\end{aligned}$$

$$+\left(\mathcal{E}^*\nabla\varphi_n-\mathcal{E}^*\nabla\varphi_n^{hk}\right)+\theta_n^{hk}\left(\mathcal{R}_n\right)\right].$$

Here, we used the symbol I for the identity application on $\mathcal{H}$. using the hypotheses on operators $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{E}$, as well as inequality $|\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\tau}|_{\mathcal{H}}\leq|\boldsymbol{\tau}|_{\mathcal{H}}$, we have

$$\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2\leq c\left[\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2\right] \tag{5.18}$$

$$+c\left[\left|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}^{hk}\right|_V^2+\left|\varphi_n-\varphi_n^{hk}\right|_W^2+\left|\theta_n^{hk}\left(\mathcal{R}_n\right)\right|_{\mathcal{H}}^2\right].$$

For $n=0$, using (4.1) at $t=0$ and (4.3), we have

$$\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h=(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_0+\mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h$$

$$=(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_0+\mathcal{P}_{\mathcal{H}^h}\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h)\right)+\left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)-\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\right)\right.$$

$$\left.+\left(\mathcal{E}^*\nabla\varphi_0-\mathcal{E}^*\nabla\varphi_0^h\right)\right].$$

Using (3.17) – (3.19) we find

$$\left|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\right|_{\mathcal{H}}^2\leq c\left[\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_0\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right|_V^2\right]. \tag{5.19}$$

$$+c\left[\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V^2+\left|\varphi_0-\varphi_0^h\right|_W^2\right]$$

We combine (4.1) and (4.2), taking $t=t_n$ for all $\boldsymbol{v}\in V$ and $n\geq 1$, we obtain

$$\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n)+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)+\int_0^{t_n}\left(\mathcal{R}_n\right)(s)ds+\mathcal{E}^*\nabla\varphi_n,\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{v}_n)\right)_{\mathcal{H}} \tag{5.20}$$

$$+j(\boldsymbol{u}_n,\boldsymbol{v},\zeta_n)-j(\boldsymbol{u}_n,\boldsymbol{v}_n,\zeta_n)\geq(\boldsymbol{f}_n,\boldsymbol{v}-\boldsymbol{v}_n)_V.$$

By combining (4.6) and (4.7) to write for all $\boldsymbol{v}^h\in V^h$ and $n\geq 1$

$$(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})+\mathcal{E}^*\nabla\varphi_n^{hk}+k\sum_{j=0}^{n-1}\left(\mathcal{R}_n\right)_j^{hk},\boldsymbol{\varepsilon}(\boldsymbol{v}^h-\boldsymbol{v}_n^{hk}))_{\mathcal{H}} \tag{5.21}$$

$$+j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}^h,\zeta_n^{hk})-j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n^{hk},\zeta_n^{hk})\geq(\boldsymbol{f}_n,\boldsymbol{v}^h-\boldsymbol{v}_n^{hk})_V.$$

From (3.17) the hypothesis on $\mathcal{A}$, we have for all $n\geq 1$

$$\begin{aligned}m_{\mathcal{A}}\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2 &\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_{\mathcal{H}}\\ &= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_{\mathcal{H}}\\ &\quad -(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h))_{\mathcal{H}}\\ &\quad +(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}^h))_{\mathcal{H}}.\end{aligned}$$

We use (5.20) with $\boldsymbol{v}=\boldsymbol{v}_n^{hk}$ to estimate the first term and (5.21) to estimate the third term, we add $(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h))_{\mathcal{H}}-(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h))_{\mathcal{H}}$ to the second side, after some elementary algebraic operations, we obtain

$$m_{\mathcal{A}}\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2 \tag{5.22}$$

$$\leq(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h))_{\mathcal{H}}+\left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)-\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})+\theta_n^{hk}\left(\mathcal{R}_n\right)\right.$$

$$\left.+\mathcal{E}^*\nabla\varphi_n-\mathcal{E}^*\nabla\varphi_n^{hk},\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h)\right)_{\mathcal{H}}-\left(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)-\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})+\theta_n^{hk}\left(\mathcal{R}_n\right)\right.$$

$$\left.+\mathcal{E}^*\nabla\varphi_n-\mathcal{E}^*\nabla\varphi_n^{hk},\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\right)_{\mathcal{H}}+j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}^h,\zeta_n^{hk})-j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n,\zeta_n^{hk})$$

$$+j(\boldsymbol{u}_n,\boldsymbol{v}_n^{hk},\zeta_n)-j(\boldsymbol{u}_n,\boldsymbol{v}_n,\zeta_n)+j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n,\zeta_n^{hk})-j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n^{hk},\zeta_n^{hk})+\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right),$$

where

$$\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)=-(\boldsymbol{\sigma}_n,\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}^h))_{\mathcal{H}}+(\boldsymbol{f}_n,\boldsymbol{v}_n-\boldsymbol{v}^h)_V. \tag{5.23}$$

From (3.30) the definition of j, we have for all $n\geq 1$

$$\left|j(\boldsymbol{u}_n,\boldsymbol{v}_n^{hk},\zeta_n)-j(\boldsymbol{u}_n,\boldsymbol{v}_n,\zeta_n)+j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n,\zeta_n^{hk})-j(\boldsymbol{u}_{n-1}^{hk},\boldsymbol{v}_n^{hk},\zeta_n^{hk})\right|$$

$$=\left|\int_{\Gamma_3}p_\nu(u_{n\nu}-g-\zeta_n\,)v_{n\nu}^{hk}da+\int_{\Gamma_3}p_\tau(u_{n\nu}-g-\zeta_n\,)\left|\boldsymbol{v}_{n\tau}^{hk}-\boldsymbol{v}^*\right|da\right.$$

$$
\begin{aligned}
&-\int_{\Gamma_3} p_\nu(u_{n\nu}-g-\zeta_n\ )\upsilon_{n\nu}da-\int_{\Gamma_3} p_\tau(u_{n\nu}-g-\zeta_n\ )\left|\boldsymbol{\upsilon}_{n\tau}-\boldsymbol{\upsilon}^*\right|da\\
&+\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon_{n\nu}da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{\upsilon}_{n\tau}-\boldsymbol{\upsilon}^*\right|da\\
&-\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon^{hk}_{n\nu}da-\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{\upsilon}^{hk}_{n\tau}-\boldsymbol{\upsilon}^*\right|da\Big|\\
&=\Big|\int_{\Gamma_3}\left[p_\nu(u_{n\nu}-g-\zeta_n\ )-p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right]\left[\upsilon^{hk}_{n\nu}-\upsilon_{n\nu}\right]da\\
&+\int_{\Gamma_3}\left[p_\tau(u_{n\nu}-g-\zeta_n\ )-p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right]\left[\left|\boldsymbol{\upsilon}^{hk}_{n\tau}-\boldsymbol{\upsilon}^*\right|-\left|\boldsymbol{\upsilon}_{n\tau}-\boldsymbol{\upsilon}^*\right|\right]da\Big|\\
&\leq\int_{\Gamma_3}\left|p_\nu(u_{n\nu}-g-\zeta_n\ )-p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right|\left|\upsilon^{hk}_{n\nu}-\upsilon_{n\nu}\right|da\\
&+\int_{\Gamma_3}\left|p_\tau(u_{n\nu}-g-\zeta_n\ )-p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\right|\left|\boldsymbol{\upsilon}^{hk}_{n\tau}-\boldsymbol{\upsilon}_{n\tau}\right|da.
\end{aligned}
$$

From (3.21) and inequality (3.12) with the inequality $|u_r|\leq|\boldsymbol{u}|$ $(r=\nu,\tau)$ $\forall\boldsymbol{u}\in\mathbb{R}^d$, we find for all $n\geq1$

$$
\begin{aligned}
&\left|j(\boldsymbol{u}_n,\boldsymbol{\upsilon}^{hk}_n,\zeta_n)-j(\boldsymbol{u}_n,\boldsymbol{\upsilon}_n,\zeta_n)+j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}_n,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}^{hk}_n,\zeta^{hk}_n)\right| \qquad (5.24)\\
&\leq(L_\nu+L_\tau)\,c_0^2\left|\boldsymbol{u}_n-\boldsymbol{u}^{hk}_{n-1}\right|_V\left|\boldsymbol{\upsilon}_n-\boldsymbol{\upsilon}^{hk}_n\right|_V\\
&+(L_\nu+L_\tau)\,c_0\left|\zeta_n-\zeta^{hk}_n\right|_{L^2(\Gamma_3)}\left|\boldsymbol{\upsilon}_n-\boldsymbol{\upsilon}^{hk}_n\right|_V.
\end{aligned}
$$

Similarly, we have for all $n\geq1$

$$
\begin{aligned}
&\left|j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}^h,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}_n,\zeta^{hk}_n)\right|\\
&=\Big|\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon^h_\nu da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{\upsilon}^h_\tau-\boldsymbol{\upsilon}^*\right|da\\
&-\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\upsilon_{n\nu}da-\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{\upsilon}_{n\tau}-\boldsymbol{\upsilon}^*\right|da\Big|\\
&\leq\int_{\Gamma_3} p_\nu(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\upsilon^h_\nu-\upsilon_{n\nu}\right|da+\int_{\Gamma_3} p_\tau(u^{hk}_{n-1\nu}-g-\zeta^{hk}_n\ )\left|\boldsymbol{\upsilon}^h_\tau-\boldsymbol{\upsilon}_{n\tau}\right|da
\end{aligned}
$$

Using (3.21) and (3.12) to deduce that

$$
\left|j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}^h,\zeta^{hk}_n)-j(\boldsymbol{u}^{hk}_{n-1},\boldsymbol{\upsilon}_n,\zeta^{hk}_n)\right|\leq(m_\nu+m_\tau)\,c_0\left|\boldsymbol{\upsilon}_n-\boldsymbol{\upsilon}^h\right|^2_V. \qquad (5.25)
$$

We substitute (5.24) – (5.25) into (5.22) and using the assumptions on $\mathcal{A}$, $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{E}$, the Cauchy-Schwarz inequality and (5.8), we obtain for all $n\geq1$

$$
\begin{aligned}
&\left|\boldsymbol{\upsilon}_n-\boldsymbol{\upsilon}^{hk}_n\right|^2_V\leq c\left(\left|\boldsymbol{u}_n-\boldsymbol{u}^{hk}_{n-1}\right|^2_V+\left|\varphi_n-\varphi^{hk}_n\right|^2_W+\left|\zeta_n-\zeta^{hk}_n\right|^2_{L^2(\Gamma_3)}\right) \qquad (5.26)\\
&+c\left(\left|\boldsymbol{\upsilon}_n-\boldsymbol{\upsilon}^h\right|^2_V+\left|\theta^{hk}_n\left(\mathcal{R}_n\right)\right|^2_{\mathcal{H}}\right)+\left|\mathcal{R}_{1,n}\left(\boldsymbol{\upsilon}^h\right)\right|.
\end{aligned}
$$

Similarly, we apply (4.1) – (4.2) at $t=0$ with the initial condition $\zeta\left(0\right)=0$, for all $\boldsymbol{\upsilon}\in V$, we find

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0)+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)+\mathcal{E}^*\nabla\varphi_0,\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}-\boldsymbol{\upsilon}_0))_{\mathcal{H}} \qquad (5.27)\\
&+j(\boldsymbol{u}_0,\boldsymbol{\upsilon},0)-j(\boldsymbol{u}_0,\boldsymbol{\upsilon}_0,0)\geq(\boldsymbol{f}\left(0\right),\boldsymbol{\upsilon}-\boldsymbol{\upsilon}_0)_V.
\end{aligned}
$$

Using (4.3) – (4.4) with $\zeta^h_0=0$ to see that for all $\boldsymbol{\upsilon}^h\in V^h$

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h_0)+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^h_0)+\mathcal{E}^*\nabla\varphi^h_0,\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h-\boldsymbol{\upsilon}^h_0))_{\mathcal{H}} \qquad (5.28)\\
&+j(\boldsymbol{u}^h_0,\boldsymbol{\upsilon}^h,0)-j(\boldsymbol{u}^h_0,\boldsymbol{\upsilon}^h_0,0)\geq(\boldsymbol{f}\left(0\right),\boldsymbol{\upsilon}^h-\boldsymbol{\upsilon}^h_0)_V.
\end{aligned}
$$

We use (3.17), we have

$$
\begin{aligned}
m_{\mathcal{A}}\left|\boldsymbol{\upsilon}_0-\boldsymbol{\upsilon}^h_0\right|^2_V &\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0)-\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0-\boldsymbol{\upsilon}^{hk}_0))_{\mathcal{H}}\\
&= (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0),\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0-\boldsymbol{\upsilon}^h_0))_{\mathcal{H}}\\
&\quad-(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}_0-\boldsymbol{\upsilon}^h))_{\mathcal{H}}\\
&\quad+(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h_0),\boldsymbol{\varepsilon}(\boldsymbol{\upsilon}^h_0-\boldsymbol{\upsilon}^h))_{\mathcal{H}}.
\end{aligned}
$$

Using (5.27) with $\boldsymbol{v} = \boldsymbol{v}_0^h$ to estimate the first term and (5.27) to estimate the third term, and adding $(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}} - (\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}}$ to the second side, we obtain

$$m_{\mathcal{A}} \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 \tag{5.29}$$

$$\leq (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_0^h), \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h))_{\mathcal{H}} + \big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^{hk})$$
$$+\mathcal{E}^*\nabla\varphi_0 - \mathcal{E}^*\nabla\varphi_0^h, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}^h)\big)_{\mathcal{H}} - \big(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)$$
$$+\mathcal{E}^*\nabla\varphi_0 - \mathcal{E}^*\nabla\varphi_0^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{v}_0 - \boldsymbol{v}_0^h)\big)_{\mathcal{H}} + j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0)$$
$$+j(\boldsymbol{u}_0, \boldsymbol{v}_0^h, 0) - j(\boldsymbol{u}_0, \boldsymbol{v}_0, 0) + j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0) + \mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right).$$

From (3.21) and by the same argument that we used in (5.24), we find

$$\left|j(\boldsymbol{u}_0, \boldsymbol{v}_0^h, 0) - j(\boldsymbol{u}_0, \boldsymbol{v}_0, 0) + j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0^h, 0)\right| \tag{5.30}$$
$$\leq (L_\nu + L_\tau)\, c_0^2 \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V.$$

Similarly, using a similar argument that we used in (5.25) to see that

$$\left|j(\boldsymbol{u}_0^h, \boldsymbol{v}^h, 0) - j(\boldsymbol{u}_0^h, \boldsymbol{v}_0, 0)\right| \leq (m_\nu + m_\tau)\, c_0 \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2. \tag{5.31}$$

We substitute (5.30) – (5.31) into (5.29) and using (3.17) – (3.19), the Cauchy-Schwarz inequality and (5.8), we obtain

$$\left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 \leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \varphi_0^h\right|_W^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2\right) + \left|\mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right)\right|. \tag{5.32}$$

Combining (5.10) and (5.19) with (5.32), it is easy to see that

$$\left|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\right|_{\mathcal{H}}^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right|_V^2 + \left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \varphi_0^h\right|_W^2 \tag{5.33}$$
$$\leq c\left(\left|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right|_V^2 + \left|\varphi_0 - \phi^h\right|_W^2 + \left|\boldsymbol{v}_0 - \boldsymbol{v}^h\right|_V^2 + \left|(I - \mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_0\right|_{\mathcal{H}}^2\right) + \left|\mathcal{R}_{1,0}\left(\boldsymbol{v}^h\right)\right|.$$

On the other hand, for the wear function, we use (3.36) at $t = t_n$, and $\zeta(0) = 0$, we obtain for all $n \geq 1$

$$\zeta_n = k_0 \upsilon^* \int_0^{t_n} p_\nu\left(u_\nu(s) - g - \zeta(s)\right) ds, \tag{5.34}$$

we subtract (4.9) from (5.34) to see that

$$\zeta_n - \zeta_n^{hk} = k_0\upsilon^* \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(p_\nu\left(u_\nu(s) - g - \zeta(s)\right) - p_\nu\left(u_{\nu j}^{hk} - g - \zeta_j^{hk}\right)\right) ds\right],$$

using (3.21), the inequality $|u_\nu| \leq |\boldsymbol{u}|\ \forall \boldsymbol{u} \in \mathbb{R}^d$ and (3.12), we obtain

$$\begin{aligned}
\left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} &\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|u_\nu(s) - u_{\nu j}^{hk}\right|_{L^2(\Gamma_3)} + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds \\
&\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j^{hk}\right|_{L^2(\Gamma_3)^d} + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds \\
&\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j^{hk}\right|_V + \left|\zeta(s) - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right] ds,
\end{aligned}$$

therefore

$$\begin{aligned}
\left|\zeta_n - \zeta_n^{hk}\right|_{L^2(\Gamma_3)} &\leq c\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\left|\boldsymbol{u}(s) - \boldsymbol{u}_j\right|_V + \left|\zeta(s) - \zeta_j\right|_{L^2(\Gamma_3)}\right] ds \\
&\quad + ck\sum_{j=0}^{n-1} \left[\left|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\right|_V + \left|\zeta_j - \zeta_j^{hk}\right|_{L^2(\Gamma_3)}\right],
\end{aligned}$$

using (5.1), the first sum can be bounded by ck where the constant c is proportional to $|\dot{\boldsymbol{u}}|_{C(0,T;V)}+\left|\dot{\zeta}\right|_{C(0,T;L^2(\Gamma_3))}$. Thus

$$\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2\le ck^2+ck\sum_{j=0}^{n-1}\left|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\right|_V^2+ck\sum_{j=0}^{n-1}\left|\zeta_j-\zeta_j^{hk}\right|_{L^2(\Gamma_3)}^2 \tag{5.35}$$

By adding (5.9), (5.11) – (5.12), (5.16), (5.18), (5.26) and (5.35) to obtain for all $n\ge 1$

$$\begin{aligned}&\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V^2+\left|\varphi_n-\varphi_n^{hk}\right|_W^2+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2\\ &\le ck^2+c\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V^2+c\left[\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2+\left|\varphi_n-\phi^h\right|_W^2+\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_V^2\right]\\ &+\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|+ck\sum_{j=0}^{n-1}\Big\{\left|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right|_V^2+\left|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\right|_V^2\\ &+\left|\varphi_j-\varphi_j^{hk}\right|_W^2+\left|\zeta_j-\zeta_j^{hk}\right|_{L^2(\Gamma_3)}^2\Big\}.\end{aligned}$$

From this inegality and (5.33), applying Gronwall's Lemma (see for example Sofonea et al., 2012) to see that

$$\max_{0\le n\le N}\left\{\begin{array}{c}\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}^2+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V^2+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V^2\\ +\left|\varphi_n-\varphi_n^{hk}\right|_W^2+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}^2\end{array}\right\} \tag{5.36}$$

$$\begin{aligned}&\le ck^2+c\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V^2+c\max_{0\le n\le N}\Big\{\left|(I-\mathcal{P}_{\mathcal{H}^h})\,\boldsymbol{\sigma}_n\right|_{\mathcal{H}}^2+\inf_{\phi^h\in W^h}\left|\varphi_n-\phi^h\right|_W^2\\ &+\inf_{\boldsymbol{v}^h\in V^h}\left[\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_V^2+\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\right]\Big\}.\end{aligned}$$

To find a bound of $\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)$ defined in (5.23), we integrate by parts the first term to obtain

$$\begin{aligned}\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)&=\int_\Omega Div\boldsymbol{\sigma}_n.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx-\int_\Gamma(\boldsymbol{\sigma\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\ &+(\boldsymbol{f}_n,\boldsymbol{v}_n-\boldsymbol{v}^h)_V.\end{aligned}$$

Using (3.28) and we apply (3.3) and (3.6) at $t=t_n$ to see that for all $n\ge 0$

$$\begin{aligned}\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)&=-\int_\Omega\boldsymbol{f}_{0n}.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx-\int_{\Gamma_2}\boldsymbol{f}_{2n}\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\ &-\int_{\Gamma_3}(\boldsymbol{\sigma\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da+\int_\Omega\boldsymbol{f}_{0n}.\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)dx\\ &+\int_{\Gamma_2}\boldsymbol{f}_{2n}\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da\\ &=-\int_{\Gamma_3}(\boldsymbol{\sigma\nu})_n\left(\boldsymbol{v}_n-\boldsymbol{v}^h\right)da,\end{aligned}$$

using the Cauchy-Schwarz inequality we see that

$$\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\le|(\boldsymbol{\sigma\nu})_n|_{L^2(\Gamma_3)^d}\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d}.$$

From (5.4) we deduce that

$$\left|\mathcal{R}_{1,n}\left(\boldsymbol{v}^h\right)\right|\le c\left|\boldsymbol{v}_n-\boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d},$$

Combining the previous estimate with (5.36), we find (5.5).□

Theorem 5.2. *Suppose that k is sufficiently small. Then, under the regularity assumptions* (5.1) – (5.4), *we have the following error estimate*

$$\max_{0\le n\le N}\left\{\begin{array}{c}\left|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\right|_{\mathcal{H}}+\left|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\right|_V+\left|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right|_V\\ +\left|\varphi_n-\varphi_n^{hk}\right|_W+\left|\zeta_n-\zeta_n^{hk}\right|_{L^2(\Gamma_3)}\end{array}\right\}\le c\,(h+k)\,. \tag{5.37}$$

Proof. Under assumptions (5.3) and (5.4), we can apply the standard theory of finite element interpolation (see for example Braess, 2007 and Sofonea et al., 2005) to see that

$$\left|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right|_V\le ch\left|\boldsymbol{u}_0\right|_{H^2(\Omega)^d},$$

$$\max_{0\le n\le N} |\boldsymbol{\sigma}_n - \mathcal{P}_{\mathcal{H}^h}\boldsymbol{\sigma}_n|_{\mathcal{H}} \le ch\,|\boldsymbol{\sigma}|_{C\left(0,T;H^1(\Omega)^{d\times d}\right)},$$

$$\max_{0\le n\le N}\inf_{\boldsymbol{v}^h\in V^h} \left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_V \le ch\,|\boldsymbol{v}|_{C(0,T;H^2(\Omega)^d)},$$

$$\max_{0\le n\le N}\inf_{\boldsymbol{v}^h\in V^h} \left|\boldsymbol{v}_n - \boldsymbol{v}^h\right|_{L^2(\Gamma_3)^d} \le ch^2\,|\boldsymbol{v}|_{C(0,T;H^2(\Gamma_3)^d)},$$

$$\max_{0\le n\le N}\inf_{\phi^h\in W^h} \left|\varphi_n - \phi^h\right|_W \le ch\,|\varphi|_{C(0,T;H^2(\Omega))}.$$

Combining the previous estimates and (5.5) it leads to (5.37).□

6. Conclusion

This paper presents a model of the quasistatic contact process between an electro-viscoelastic body and a foundation. The contact was modeled by normal compliance with wear. The proof of the existence of a unique weak solution to the model has been obtained by using arguments on elliptic variational inequalities. A fully discrete scheme is used to approach the problem and an optimal order error estimate. A numerical algorithm which combines the backward Euler difference method with the finite elements method. Finally, it may be interesting to incorporate control mechanisms into the model and study the related optimal control problem. Also, the problem is relatively easy to set experimentally, and it may provide an effective way to determine some of the constants associated with the contact process, to be used in more complex physical settings.

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Declaration

The author declares no conflicts of interest.

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