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# The Impact of Imperfect Vaccination on Infectious Disease Transmission in an Age-Structured Population

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abstract

*In this paper, we consider the influence of imperfect vaccination on the spread of infectious diseases in an age-structured population. The benefits of vaccination, even if not perfect, generally outweigh the risks of severe diseases. In a mathematical system, we consider the compartment of susceptible $s$, vaccinated $v$ and infected $i$ individuals with an age structure.*
*The proposed model is globally analyzed by introducing total trajectories and employing a suitable Lyapunov functional. To illustrate our theoretical findings, we include numerical simulations at the end of the paper.*

keywords

*Age structured model; Lyapunov functional; Uniform persistence; Total trajectories; $\alpha$ and $\omega$ limit sets.*

2020 Mathematics Subject Classification

*35Q92, 37N25, 92D30*

## 1. Introduction

Since its earliest applications, vaccination has been a highly effective strategy in preventing and controlling the spread of infectious diseases. This medical intervention not only plays a crucial role in individual and collective protection by stimulating the immune system against potentially devastating pathogens but has also been a subject of in-depth research, notably through mathematical modeling. This research aims to understand its impact on the dynamics of disease transmission within populations.

The importance of vaccination in containing the spread of infectious diseases cannot be overstated. Diseases like measles, polio, and influenza historically caused widespread devastation before the advent of vaccination programs. A notable success is the eradication of smallpox in the 1980s through a coordinated global vaccination campaign. These successes demonstrate that vaccination not only prevents disease in vaccinated individuals but also interrupts the chain of transmission, providing protection to unvaccinated populations, including vulnerable individuals who cannot be vaccinated for medical reasons.

The use of mathematical models in vaccination research has proved to be a valuable tool. It helps anticipate patterns of disease spread, assess the impact of vaccination campaigns, and design informed public health policies, we cite for example papers Adimy et al., 2022; Castillo-Chávez et al., 1989; Diekmann and Heesterbeek, 2000; Ismail and Touaoula, 2018.

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While vaccines are recognized for reducing or eliminating infection rates, it's crucial to acknowledge that not all vaccines achieve 100% effectiveness Galazka et al., 1995; Grenfell and Anderson, 1989; Hethcote, 2000; Janaszek et al., 2003; Mossong et al., 2000; Scherer and McLean, 2002. Recent clinical studies have focused on understanding the impact of imperfect vaccines, characterized by waning or incomplete immunity, in controlling infectious disease transmission. These studies aim to answer key questions, including the proportion of susceptible individuals requiring immunization, the consequences of incomplete vaccine protection, and the significance of vaccine-induced immunity waning over time.

Vaccination programs offer both direct and indirect protection against infectious diseases **1**; Cai et al., 2013, 2017; Feng et al., 2020. Direct protection lowers the risk of infection in vaccinated individuals, while indirect protection limits transmission within populations. Various vaccine models, encompassing perfect and imperfect vaccines, have been explored, including all-or-nothing, leaky, and waning vaccines Mclean and Blower, 1993.

Numerous investigations have independently explored models of all-or-nothing, leaky, and waning vaccines for specific diseases. For example, Kanaan et al. devised a framework to examine the effectiveness of waning pertussis vaccines, demonstrating the potential to make inferences regarding the diminishing effects of these vaccines Kanaan and Farrington, 2002. Shim et al. employed dynamic epidemiological models for both all-or-nothing and leaky vaccines, emphasizing the critical role of accurately parameterizing vaccine effectiveness for robust model predictions Shim and Galvani, 2012. In a comprehensive study, Magpantay et al. delved into all-or-nothing, leaky, and waning vaccine models, investigating the variations in disease outcomes attributable to these different vaccine types Magpantay et al., 2014.

Studies have individually investigated these models for diseases like pertussis, measles, and rubella, with researchers assessing efficacy using dynamic epidemiological models. However, amidst the ongoing COVID-19 pandemic, vaccine prioritization discussions have primarily focused on all-or-nothing and leaky vaccine models, neglecting the consideration of waning vaccines Bubar et al., 2020; Buckner et al., 2020; Magpantay et al., 2014. This underscores the evolving challenges and the need for a comprehensive understanding of vaccine dynamics in the current global health landscape.

Imperfect vaccination can manifest in various forms, and here are some common types:

**Partial Immunity**: Some individuals may develop partial immunity after vaccination, meaning they are not entirely protected against the disease, but the severity of the infection can be reduced.

**Limited Duration of Immunity**: In some cases, immunity acquired through vaccination may decrease over time, eventually requiring regular boosters to maintain adequate protection.

**Variable Effectiveness**: The effectiveness of a vaccine can vary based on various factors such as age, the individual's overall health, and adherence to recommended vaccination schedules.

**Protection Against Certain Serotypes**: Some vaccines may offer protection against certain serotypes of pathogens, but not all. This can lead to infections by uncovered strains.

**Rare Risks of Post-Vaccination Infection**: While vaccines are designed to prevent infections, there may be rare cases where a vaccinated person still contracts the disease. However, the severity of the infection is often reduced in these individuals.

**Viral Adaptation**: Some viruses can undergo mutations over time, potentially reducing the effectiveness of vaccines against emerging strains. This may require regular adjustments to vaccine formulations.

**Variable Immune Responses**: Individuals may have different immune responses to vaccination due to genetic or environmental factors, leading to varying levels of protection.

It is important to note that, despite these imperfections, vaccination remains an essential tool for preventing and controlling the spread of infectious diseases. The benefits of vaccination, even if not perfect, generally outweigh the risks of severe diseases.

In Hathout et al., 2022, we considered a protective compartment, incorporating various aspects such as vaccination, within an $SI$ model. The model construction implies that the protection (or vaccination) was perfect, as there was no transition of protected individuals to the infected compartment (perfect protection). The results are obtained based on the basic reproduction number $\mathcal{R}_0$. In this work, we will retain the same model while introducing an interaction between the i-class and v-class of the vaccinated. This implies that

vaccination is not perfect:

$$\begin{cases} \dfrac{\partial s(t,a)}{\partial t} + \dfrac{\partial s(t,a)}{\partial a} = -(\mu_s(a) + \delta(a))s(t,a) - \beta_s(a)s(t,a)J(t), & t > 0, \\ \dfrac{\partial v(t,a)}{\partial t} + \dfrac{\partial v(t,a)}{\partial a} = -(\mu_v(a) + k(a))v(t,a) - \beta_v(a)v(t,a)J(t), & t > 0, \quad a > 0, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -(\mu_i(a) + q(a))i(t,a), & t > 0, \quad a > 0, \\ s(t,0) = A + (1-\rho)\displaystyle\int_0^\infty k(a)v(t,a)da, & t > 0, \\ v(t,0) = \displaystyle\int_0^\infty \delta(a)s(t,a)da + \rho\int_0^\infty k(a)v(t,a)da, & t > 0, \\ i(t,0) = J(t)\displaystyle\int_0^\infty (\beta_s(a)s(t,a) + \beta_v(a)v(t,a))\,da, & t > 0, \\ J(t) = \displaystyle\int_0^\infty \theta(a)i(t,a)da, \end{cases} \tag{1}$$

with initial conditions:

$$s(0,.) = \bar{s}(.) \in L^1_+(\mathbb{R}^+), \quad v(0,\cdot) = \bar{v}(\cdot) \in L^1_+(\mathbb{R}^+), \quad i(0,.) = \bar{i}(\cdot) \in L^1_+(\mathbb{R}^+).$$

$s(t,a)$, $v(t,a)$ and $i(t,a)$ are respectively the population densities of susceptible, protected and infected individuals, at time $t$ with age $a$. Here, the age represents the time spent in each class. The functions $\mu_s(\cdot), \mu_v(\cdot)$

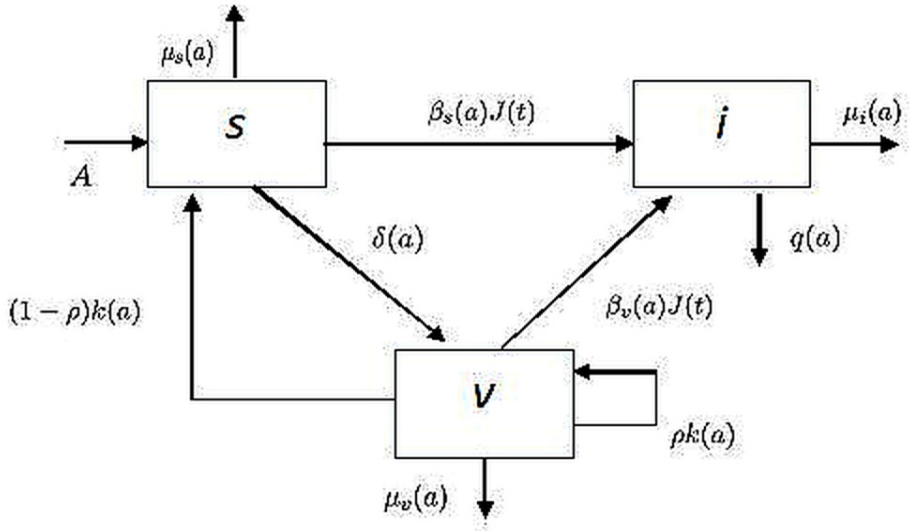


Figure 1: Diagram flux of the system (1).

and $\mu_i(\cdot)$ are the age-dependent per capita death for susceptible, infected, protected populations, respectively, with age $a$. The parameter $\rho$ is the probability of returning again to the v-class, more precisely, it is the specific protection rate which highlights the concerned persons for the re-protection. The constant $A$ represents the entering flux into the s-class. The functions $\delta(\cdot), k(\cdot)$ are, respectively, the protection rate, removing from v-class to s-class (rate of losing protection) and $\beta_s(\cdot), \beta_v(\cdot)$ are transmission rates. $q(.)$ is the recovering rate after spending $a$ time in i-class, $\theta(.)$ represents the infectivity rate for an arbitrary infected person. See Figure 1.

We organize this research in the following form : After presenting the assumptions about the model data in the Preliminaries section, we will provide Volterra formulation for problem (1).We ensure the existence of a global compact attractor in section 4. Section 5 will offer the system of total trajectories that will enable us to study the global stability of solutions. Subsection 6.1 is dedicated to demonstrating the global stability of the

trivial equilibrium (which always exists) in the case of $\mathcal{R}_0 \leq 1$. In subsection 6.2, we will discuss the emergence of the positive (endemic) equilibrium and the persistence of the disease, where $\mathcal{R}_0 > 1$, as well as its global stability through an appropriate Lyapunov function. Numerical simulation plays a crucial role in validating and illustrating theoretical findings in such studies. For example, in a recently published paper Benchaira et al., 2024, the authors in this paper show, by simulation, that the newly proposed estimator behaves well both in terms of bias and mean squared error. Similarly, in our work, theoretical results will be confirmed through numerical simulation examples, highlighting the interplay between theory and computation in epidemiological research. **Note that the theorems and proofs stated in Hathout et al., 2022 will not be reiterated here**.

## 2. Preliminaries

We assume that:

($\mathbf{H_1}$) All the parameters mentioned in model (1) are assumed to be positive, we also assume that $\mu_s, \mu_v, \mu_i, q, \theta \in L^{\infty}_{+}(\mathbb{R}^{+}) \setminus \{0_{L^{\infty}}\}$. In addition

$$M := \min\left\{ess \inf_{a\in\mathbb{R}_+}\{\mu_s(a)\}, ess \inf_{a\in\mathbb{R}_+}\{\mu_v(a)\}, ess \inf_{a\in\mathbb{R}_+}\{\mu_i(a)\}\right\} > 0.$$

($\mathbf{H_2}$) $\delta, k, \beta_s$ and $\beta_v$ are positive, Lipschitz continuous functions in $\mathbb{R}^{+}$, with $\beta_s, \beta_v \in L^1(\mathbb{R}^{+}) \cap L^{\infty}(\mathbb{R}^{+})$.

In addition, we set the functional space for system (1)

$$X_+ := L^1_+\left(\mathbb{R}^+\right) \times L^1_+\left(\mathbb{R}^+\right) \times L^1_+\left(\mathbb{R}^+\right),$$

which is the positive cone of

$$X := L^1\left(\mathbb{R}^+\right) \times L^1\left(\mathbb{R}^+\right) \times L^1\left(\mathbb{R}^+\right),$$

equipped with the norm

$$\left\|\left(s(t,.), v(t,.), i(t,.)\right)\right\|_X = \int_0^{\infty} |s(t,a)|\, da + \int_0^{\infty} |v(t,a)|\, da + \int_0^{\infty} |i(t,a)|\, da.$$

The following theorem guarantees existence and uniqueness of solutions for (1):

**Theorem 2.1.** *Let $x_0 = (\bar{s}(.), \bar{v}(.), \bar{i}(.)) \in X_+$, then there exists a unique nonnegative solution $(s(.), v(.), i(.)) \in C(\mathbb{R}^+, L^1(\mathbb{R}^+)) \times C(\mathbb{R}^+, L^1(\mathbb{R}^+)) \times C(\mathbb{R}^+, L^1(\mathbb{R}^+))$ to system (1).*

*Proof.* By applying the Banach fixed point method we can demonstrate existence and uniqueness of the nonnegative solution to (1) for any positive initial condition. This procedure is used in Bentout and Touaoula, 2015 and could be applied here. □

## 3. Volterra integral equation

By the characteristics method Webb, 1985, the PDE's system (1) can be expressed by Volterra equation as the following:

$$s(t,a) = \begin{cases} s(t-a,0)\pi_s(a)\Gamma_s(t-a,a), & t > a \geq 0, \\ \bar{s}(a-t)\frac{\pi_s(a)\Gamma_s(t-a,a)}{\pi_s(a-t)\Gamma_s(t-a,a-t)}, & a \geq t \geq 0, \end{cases}$$

with

$$\Gamma_s(t,a) = exp\left\{-\int_0^a \beta_s(\sigma)J(t+\sigma)d\sigma\right\}, \qquad \pi_s(a) = exp\left\{-\int_0^a (\mu_s(\sigma)+\delta(\sigma))\, d\sigma\right\},$$

$$v(t,a) = \begin{cases} v(t-a,0)\pi_v(a)\Gamma_v(t-a,a), & t > a \geq 0, \\ \bar{v}(a-t)\frac{\pi_v(a)\Gamma_v(t-a,a)}{\pi_v(a-t)\Gamma_v(t-a,a-t)}, & a \geq t \geq 0, \end{cases}$$

where

$$\Gamma_v(t,a) = exp\left\{-\int_0^a \beta_v(\sigma)J(t+\sigma)d\sigma\right\}, \qquad \pi_v(a) = exp\left\{-\int_0^a (\mu_v(\sigma)+k(\sigma))d\sigma\right\}.$$

We also have

$$i(t,a) = \begin{cases} i(t-a,0)\pi_i(a), & t > a \geq 0, \\ \bar{i}(a-t)\frac{\pi_i(a)}{\pi_i(a-t)}, & a \geq t \geq 0, \end{cases}$$

with

$$\pi_i(a) = exp\left\{-\int_0^a (\mu_i(\sigma) + q(\sigma))d\sigma\right\}.$$

## 4. Global compact attractor

In the subsequent we set

$$\mathbb{E}_M = \left\{(s,v,i) \in X_+ : \int_0^\infty s\,(t,a)\,da + \int_0^\infty v\,(t,a)\,da + \int_0^\infty i\,(t,a)\,da \leq \frac{A}{M}\right\}.$$

It is not difficult to show that $\lambda$ is positively invariant. We can also prove that there exists a continuous semi-flow $\Phi(t,x_0) = \Phi_t(x_0)$ such that $\Phi_t(x_0) = (s(t,.), v(t,.), i(t,.))$ with $t \in \mathbb{R}^+, x_0 \in X_+$, where $(s,v,i)$ is solution of (1). Extracting some properties on the set $\mathbb{E}_M$ and the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ as well by the following proposition:

**Proposition 4.1.** *Hathout et al., 2022 Let $\Phi_t$ be the semi-flow of the system (1), then we have the following aspects*

**(i)** *$\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ is point dissipative. Further, $\mathbb{E}_M$ attracts all point in $X_+$*

**(ii)** *$\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+} \in \mathbb{E}_M$ for all $t \geqslant 0$ and $x_0 \in \mathbb{E}_M$.*

**Theorem 4.2.** *Hathout et al., 2022 The semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ engendered by system (1) is asymptotically smoothMagal and Thieme, 2004; Magal and Zhao, 2005. In addition, $\Phi_t(x_0)$ has a compact attractor $\boldsymbol{B}$ restrained to $X_+$. Moreover $\boldsymbol{B}$ attracts all bounded sets of $X_+$.*

## 5. Total trajectories

A total trajectory is a function $\phi$ that satisfies $\phi(t+r) = \Phi(t,\phi(r))$ for all $t \in \mathbb{R}$ and $r \geq 0$. Thus, for $\phi(t) = (s(t,.), v(t,.), i(t,.))$, $t \in \mathbb{R}$ and $a \geq 0$, we define a total trajectory as

$$\begin{cases} s(t,a) = s(t-a,0)\pi_s(a)\Gamma_s(t-a,a), \\ \Gamma_s(t,a) = exp\left\{-\int_0^a \beta_s(\sigma)J(t+\sigma)d\sigma\right\}, \\ v(t,a) = v(t-a,0)\pi_v(a)\Gamma_v(t-a,a), \\ \Gamma_v(t,a) = exp\left\{-\int_0^a \beta_v(\sigma)J(t+\sigma)d\sigma\right\}, \\ i(t,a) = i(t-a,0)\pi_i(a), \\ J(t) = \int_0^\infty \theta(a)i(t,a)da, \end{cases}$$

where $s(t,0)$, $v(t,0)$ and $i(t,0)$ are defined in (1).

**Lemma 5.1.** *Hathout et al., 2022 For all $x_0 := (\bar{s}(.), \bar{v}(.), \bar{i}(.)) \in \boldsymbol{B}$, the following estimates hold true.*

$$\int_0^\infty s(t,a)da + \int_0^\infty v(t,a)da + \int_0^\infty i(t,a)da \leq \frac{A}{M}.$$

$$J(t) \leq ||\theta||_\infty \frac{A}{M},$$

*for all $t \in \mathbb{R}$ and there exist positive constants $c_1$ and $c_2$ such that*

$$s(t,a) \geq c_1\pi_s(a),$$
$$v(t,a) \geq c_2\pi_v(a),$$

*for all $t \in \mathbb{R}$ and $a \geq 0$.*

# 6. Equilibria

## 6.1. Disease free equilibrium

In this section we prove that model (1) has always the trivial equilibrium which coincides with that of Hathout et al., 2022. Then, we can use the same arguments to prove the existence as well as the global stability of this state:

**Theorem 6.1.** *The disease free equilibrium is defined by* $E_0 = (s_0(a), p_0(a), 0)$, *where*

$$\begin{cases} s_0(a) = s_0(0)\pi_s(a), \\ p_0(a) = p_0(0)\pi_v(a), \qquad a > 0, \end{cases}$$

$$\textit{with} \quad \begin{cases} s_0(0) = A + (1-\rho)p_0(0)\int_0^\infty k(a)\pi_v(a)da, \\ p_0(0) = \dfrac{A\int_0^\infty \delta(a)\pi_s(a)da}{1 - \int_0^\infty k(a)\pi_v(a)da((1-\rho)\int_0^\infty \delta(a)\pi_s(a)da + \rho)}. \end{cases}$$

Hence, we can define the basic reproduction rate $\mathcal{R}_0$ for model (1) by:

$$\mathcal{R}_0 = \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty \left(\beta_s(a)s_0(a) + \beta_v(a)v_0(a)\right) da.$$

Remark that

$$\mathcal{R}_0 = \tilde{\mathcal{R}}_0 + \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty \beta_v(a)v_0(a)da.$$

where

$$\tilde{\mathcal{R}}_0 = \int_0^\infty \theta(a)\pi_i(a)da \int_0^\infty \beta_s(a)s_0(a)da$$

is the basic reproduction rate of model 1.1 in Hathout et al., 2022 and so,

$$\mathcal{R}_0 > \tilde{\mathcal{R}}_0$$

In a model of imperfect vaccination, vaccinated individuals can still become infected, allowing the infection to spread among them. This increases the proportion of the susceptible population, leading to a higher $\mathcal{R}_0$ compared to a perfect vaccination model, where vaccinated individuals are fully protected. While vaccination reduces the likelihood of infection, the persistence of susceptibility within the vaccinated population limits its control effect. Therefore, higher vaccination rates are required to control the infection in the case of imperfect vaccination.

**Theorem 6.2.** *Assume that* $\mathcal{R}_0 \leq 1$. *The disease free equilibrium* $E_0$ *is globally stable in* $X_+$.

## 6.2. Endemic equilibrium

The main objective of this section is to show the existence and global stability of the endemic equilibrium in the case where $\mathcal{R}_0 > 1$.

### 6.2.1 Existence

In this subsection our focus is on analyzing the existence of positive endemic equilibrium for model (1). This state verifies the following system:

$$\begin{cases} \dfrac{ds^*(a)}{da} = -(\mu_s(a)+\delta(a))s^*(a) - \beta_s(a)s^*(a)J^*, \\ \dfrac{dv^*(a)}{da} = -(\mu_v(a)+k(a))v^*(a) - \beta_v(a)v^*(a)J^*, \qquad a>0, \\ \dfrac{di^*(a)}{da} = -(\mu_i(a)+q(a))i^*(a), \qquad a>0, \\ s^*(0) = A + (1-\rho)\displaystyle\int_0^\infty k(a)v^*(a)da, \\ v^*(0) = \displaystyle\int_0^\infty \delta(a)s^*(a)da + \rho\int_0^\infty k(a)v^*(a)da, \\ i^*(0) = J^*\displaystyle\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da, \\ J^* = \displaystyle\int_0^\infty \theta(a)i^*(a)da, \end{cases} \tag{2}$$

which has the solution

$$\begin{cases} s^*(a) = s^*(0)\pi_s(a)e^{-J^*\int_0^a \beta_s(\sigma)d\sigma}, \\ v^*(a) = v^*(0)\pi_v(a)e^{-J^*\int_0^a \beta_v(\sigma)d\sigma}, \\ i^*(a) = i^*(0)\pi_i(a), \qquad a>0. \end{cases} \tag{3}$$

**Theorem 6.3.** *If $\mathcal{R}_0 > 1$, there exists the unique positive equilibrium denoted $E^* = (s^*(a), v^*(a), i^*(a))$.*

*Proof.* Firstly, using the equations of (2) and (3) we have

$$\begin{aligned} v^*(0) &= \int_0^\infty \delta(a)s^*(a)da + \rho\int_0^\infty k(a)v^*(a)da \\ &= s^*(0)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a\beta_s(\sigma)d\sigma}da + \rho v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}da \\ &= \left(A+(1-\rho)v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}da\right)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a\beta_s(\sigma)d\sigma}da \\ &+ \rho v^*(0)\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}da, \end{aligned}$$

and thus

$$v^*(0) = \frac{A\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a\beta_s(\sigma)d\sigma}da}{1-\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}da\left((1-\rho)\int_0^\infty \delta(a)\pi_s(a)e^{-J^*\int_0^a\beta_s(\sigma)d\sigma}da+\rho\right)} \tag{4}$$

Next, suppose that $i^*(0) > 0$. Using the expression of $J^*$ in (2) and dividing the following equation $i^*(0) = J^*\displaystyle\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da$ by $i^*(0)$ we obtain

$$1 = \int_0^\infty \theta(a)\pi_i(a)da\int_0^\infty \left(\beta_s(a)s^*(0)\pi_s(a)e^{-J^*\int_0^a\beta_s(\sigma)d\sigma} + \beta_v(a)v^*(0)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}\right)da.$$

By employing the expression of $s^*(0)$ in (2), the last equation becomes:

$$1 = \tilde{\theta}\int_0^\infty \left(\beta_s(a)\left(A+(1-\rho)\tilde{k}v^*(0)\right)\pi_s(a)e^{-i^*(0)\tilde{\theta}\tilde{\beta}_s(a)} + \beta_v(a)v^*(0)\pi_v(a)e^{-i^*(0)\tilde{\theta}\tilde{\beta}_v(a)}\right)da, \tag{5}$$

where $\tilde{k} = \displaystyle\int_0^\infty k(a)\pi_v(a)e^{-J^*\int_0^a\beta_v(\sigma)d\sigma}da$; $\tilde{\theta} = \displaystyle\int_0^\infty \theta(a)\pi_i(a)da$ and $\tilde{\beta}_{s,v}(a) = \displaystyle\int_0^a \beta_{s,v}(\sigma)d\sigma$.

Now, using the expression of $v^*(0)$ in (4) and the fact that $J^* = i^*(0)\tilde{\theta}$, we can rewrite problem (5) as the following :

$$1 = F(i^*(0)),$$

We can easily prove that $F$ is a decreasing function. Furthermore, observe that $F(0) = \mathcal{R}_0$ and $\lim\limits_{y\to+\infty} F(y) = 0$. Therefore, problem (5) has a unique positive solution if $\mathcal{R}_0 > 1$. The proof is reached. □

### 6.2.2 Main results of uniform persistence

For the purpose of the well posedness of the Lyapunov function obtained in the next section we will show the persistence result. So, we define the following sets:

$$X^0 = \left\{ (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot)) \in X_+; \quad \int_0^\infty \theta(a)\bar{i}(a)da > 0 \right\}$$

$$\partial X^0 = \left\{ (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot)) \in X_+; \quad \int_0^\infty \theta(a)\bar{i}(a)da = 0 \right\}.$$

So, we write $X_+ = X^0 \cup \partial X^0$. For $x_0 = (\bar{s}(\cdot), \bar{v}(\cdot), \bar{i}(\cdot))$, we also denote

$$M_\partial = \{x_0 \in \partial X^0; \quad \Phi_t(x_0) \in \partial X^0, \quad \text{for all } t \geq 0\}.$$

We have the following theorems:

**Lemma 6.4.** *The subset $X^0$ is positively invariant under the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$. Furthermore, the disease free equilibrium is globally asymptotically stable for the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ restricted to $M_\partial$.*

**Theorem 6.5.** *Smith and Zhao, 2001 Assume that $\mathcal{R}_0 > 1$, the semi-flow $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ is uniformly persistent with respect to $(X^0, \partial X^0)$, i.e., there exists $\epsilon > 0$ which is independent of initial values such that $\liminf_{t\to\infty} \int_0^\infty \theta(a)i(t,a)da \geq \epsilon$ for all $x_0 \in X^0$. Moreover, there exists a compact subset $\boldsymbol{B}_0$ of $X^0$ which is a global attractor for $\{\Phi_t(x_0)\}_{t\in\mathbb{R}^+}$ in $X^0$.*

**Lemma 6.6.** *For all $x_0 \in \boldsymbol{B}_0, a > 0$ and $t \in \mathbb{R}$, there exist positive constant $c$ such that:*

$$\frac{s(t,a)}{s^*(a)} > c, \qquad \frac{v(t,a)}{v^*(a)} > c, \qquad \frac{i(t,a)}{i^*(a)} > c.$$

### 6.2.3 Global stability

**Theorem 6.7.** *Assume that $\mathcal{R}_0 > 1$. The endemic equilibrium is globally stable in $\boldsymbol{B}_0 \subset X^0$.*

*Proof.* We define $H(x) = x - ln(x) - 1$ and consider the following Lyapunov functional:

$$W(t) = \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s(a)da + \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v(a)da + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i(a)da$$

with $\quad \phi_s(a) = \dfrac{s^*(a)}{i^*(0)}, \qquad \phi_v(a) = \dfrac{v^*(a)}{i^*(0)}, \qquad \phi_i(a) = \dfrac{\int_a^\infty \theta(s)i^*(s)ds}{\int_0^\infty \theta(a)i^*(a)da}, \qquad a \geq 0$

Note that the functions $\phi_s$, $\phi_v$ and $\phi_i$ verify the following problems:

$$\begin{cases} \phi_s'(a) = -(\mu_s(a) + \delta(a) + \beta_s(a)J^*)\dfrac{s^*(a)}{i^*(0)}, \\ \phi_s(0) = \dfrac{s^*(0)}{i^*(0)}, \end{cases} \qquad \begin{cases} \phi_v'(a) = -(\mu_v(a) + k(a) + \beta_v(a)J^*)\dfrac{v^*(a)}{i^*(0)} \\ \phi_v(0) = \dfrac{v^*(0)}{i^*(0)}, \end{cases}$$

$$\begin{cases} \phi_i'(a) = -\dfrac{\theta(a)i^*(a)}{\int_0^\infty \theta(a)i^*(a)da}, \\ \phi_i(0) = 1 \end{cases}$$

Set

$$W_s(t) := \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s(a)da, \quad W_v(t) := \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v(a)da,$$

$$W_i(t) := \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i(a)da.$$

Using Lemma 3.3 in Hathout et al., 2022, we obtain

$$\begin{aligned} W_s'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da - J(t)\int_0^\infty \beta_s(a)\phi_s(a)\frac{s(t,a)}{s^*(a)}H'\left(\frac{s(t,a)}{s^*(a)}\right)da \\ &+ J^*\int_0^\infty \beta_s(a)\phi_s(a)\frac{s(t,a)}{s^*(a)}H'\left(\frac{s(t,a)}{s^*(a)}\right)da \end{aligned}$$

$$\begin{aligned} W_v'(t) &= \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) + \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da - J(t)\int_0^\infty \beta_v(a)\phi_v(a)\frac{v(t,a)}{v^*(a)}H'\left(\frac{v(t,a)}{v^*(a)}\right)da \\ &+ J^*\int_0^\infty \beta_v(a)\phi_v(a)\frac{v(t,a)}{v^*(a)}H'\left(\frac{v(t,a)}{v^*(a)}\right)da \end{aligned}$$

$$W_i'(t) = \phi_i(0)H\left(\frac{i(t,0)}{i^*(0)}\right) + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i'(a)da$$

Using the fact that $H'(x) = 1 - \dfrac{1}{x}$, we get

$$\begin{aligned} W_s'(t) + W_v'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da + \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) \\ &+ J^*\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da \\ &- J^*\int_0^\infty \left(\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a)\right)da + J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a)\right)da \\ &- J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da, \\ &+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da \end{aligned}$$

In addition, we have

$$J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a)\right)da = \frac{J(t)}{J^*}\frac{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da} = \frac{J(t)}{J^*}.$$

$$J(t)\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da = \frac{J(t)}{J^*}\frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da},$$

$$J^*\int_0^\infty \left(\beta_s(a)\phi_s(a) + \beta_v(a)\phi_v(a)\right)da = \frac{J^*}{J^*}\frac{\int_0^\infty \beta_s(a)s^*(a)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da} = 1,$$

$$J^*\int_0^\infty \left(\beta_s(a)\phi_s(a)\left(\frac{s(t,a)}{s^*(a)}\right) + \beta_v(a)\phi_v(a)\left(\frac{v(t,a)}{v^*(a)}\right)\right)da = \frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da}.$$

Since $W' = W_s' + W_v' + W_i'$ and

$$\begin{aligned} H\left(\frac{i(t,0)}{i^*(0)}\right) &= H\left(\frac{J(t)}{J^*}\frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da}\right) \\ &+ \frac{J(t)}{J^*}\frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \beta_s(a)s^*(a)da} \\ &- \ln\frac{J(t)}{J^*} - \ln\frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \beta_s(a)s^*(a)da} - 1, \end{aligned}$$

it follows that

$$\begin{aligned} W'(t) &= \phi_s(0)H\left(\frac{s(t,0)}{s^*(0)}\right) + \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\phi_s'(a)da + \phi_v(0)H\left(\frac{v(t,0)}{v^*(0)}\right) \\ &+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\phi_v'(a)da + \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\phi_i'(a)da \\ &+ H\left(\frac{\int_0^\infty \left(\beta_s(a)s(t,a) + \beta_v(a)v(t,a)\right)da}{\int_0^\infty \left(\beta_s(a)s^*(a) + \beta_v(a)v^*(a)\right)da}\right) + H\left(\frac{J(t)}{J^*}\right) \end{aligned}$$

On the other hand,

$$H\left(\frac{s(t,0)}{s^*(0)}\right) = H\left(\frac{A}{s^*(0)}\cdot 1 + \frac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)}\frac{\int_0^\infty k(a)v(t,a)da}{\int_0^\infty k(a)v^*(a)da}\right).$$

Since $H$ is convex and $\dfrac{A}{s^*(0)} + \dfrac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)} = 1$ then,

$$H\left(\frac{s(t,0)}{s^*(0)}\right) \leq \frac{A}{s^*(0)}\underbrace{H(1)}_{=0} + \frac{(1-\rho)\int_0^\infty k(a)v^*(a)da}{s^*(0)} H\left(\frac{\int_0^\infty k(a)v^*(a)\dfrac{v(t,a)}{v^*(a)}da}{\int_0^\infty k(a)v^*(a)da}\right).$$

By Jensen inequality, this last inequality leads to

$$H\left(\frac{s(t,0)}{s^*(0)}\right) \leq \frac{(1-\rho)}{s^*(0)}\int_0^\infty k(a)v^*(a)H\left(\frac{v(t,a)}{v^*(a)}\right)da.$$

$$\begin{aligned}
H\left(\frac{v(t,0)}{v^*(0)}\right) &= H\left(\frac{1}{v^*(0)}\left(\int_0^\infty \delta(a)s(t,a)da + \rho\int_0^\infty k(a)v(t,a)da\right)\right),\\
&= H\left(\frac{\int_0^\infty \delta(a)s^*(a)da}{v^*(0)}\frac{\int_0^\infty \delta(a)s(t,a)da}{\int_0^\infty \delta(a)s^*(a)da} + \rho\frac{\int_0^\infty k(a)v^*(a)da}{v^*(0)}\frac{\int_0^\infty k(a)v(t,a)da}{\int_0^\infty k(a)v^*(a)da}\right),\\
&\leq \frac{\int_0^\infty \delta(a)s^*(a)da}{v^*(0)} H\left(\frac{\int_0^\infty \delta(a)s^*(a)\dfrac{s(t,a)}{s^*(a)}da}{\int_0^\infty \delta(a)v^*(a)da}\right)\\
&+ \rho\frac{\int_0^\infty k(a)v^*(a)da}{v^*(0)} H\left(\frac{\int_0^\infty k(a)v^*(a)\dfrac{v(t,a)}{v^*(a)}da}{\int_0^\infty k(a)v^*(a)da}\right),\\
&\leq \frac{1}{v^*(0)}\int_0^\infty \delta(a)s^*(a)H\left(\frac{s(t,a)}{s^*(a)}\right)da + \frac{\rho}{v^*(0)}\int_0^\infty k(a)v^*(a)H\left(\frac{v(t,a)}{v^*(a)}\right)da.
\end{aligned}$$

$$\begin{aligned}
H\left(\frac{J(t)}{J^*}\right) = H\left(\frac{\int_0^\infty \theta(a)i(t,a)da}{\int_0^\infty \theta(a)i^*(a)da}\right) &= H\left(\frac{\int_0^\infty \theta(a)i^*(a)\dfrac{i(t,a)}{i^*(a)}da}{\int_0^\infty \theta(a)i^*(a)da}\right)\\
&\leq \frac{\int_0^\infty \theta(a)i^*(a)H\left(\dfrac{i(t,a)}{i^*(a)}\right)da}{\int_0^\infty \theta(a)i^*(a)da}.
\end{aligned}$$

$$\begin{aligned}
H\left(\frac{\int_0^\infty (\beta_s(a)s(t,a)+\beta_v(a)v(t,a))\,da}{\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}\right) &\leq \frac{\int_0^\infty \beta_s(a)s^*(a)H\left(\dfrac{s(t,a)}{s^*(a)}\right)da}{\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}\\
&+ \frac{\int_0^\infty \beta_v(a)v^*(a)H\left(\dfrac{v(t,a)}{v^*(a)}\right)da}{\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da}
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
W'(t) &\leq \int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\left(\phi_s'(a) + \frac{\beta_s(a)s^*(a)}{\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da} + \frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a)\right)da\\
&+ \int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\left(\phi_v'(a) + \frac{\beta_v(a)v^*(a)}{\int_0^\infty (\beta_s(a)s^*(a)+\beta_v(a)v^*(a))\,da} + \tau k(a)v^*(a)\right)da\\
&+ \int_0^\infty H\left(\frac{i(t,a)}{i^*(a)}\right)\underbrace{\left(\phi_i'(a) + \frac{\theta(a)i^*(a)}{\int_0^\infty \theta(a)i^*(a)da}\right)}_{=0}da.
\end{aligned}$$

where

$$\tau = \rho\frac{\phi_v(0)}{v^*(0)} + (1-\rho)\frac{\phi_s(0)}{s^*(0)} = \rho\frac{1}{v^*(0)}\frac{v^*(0)}{i^*(0)} + (1-\rho)\frac{1}{i^*(0)} = \frac{1}{i^*(0)},$$

Let

$$L_s(a) := \phi_s'(a) + \frac{\beta_s(a)s^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a)$$

Replacing $\phi_s'$ by its expression we get:

$$\begin{aligned} L(a) &= -\frac{s^*(a)}{i^*(0)}\left(\mu_s(a) + \delta(a) + \beta_s(a)J^*\right) + \frac{\beta_s(a)s^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{\phi_v(0)}{v^*(0)}\delta(a)s^*(a), \\ &= -\frac{1}{i^*(0)}\mu_s(a)s^*(a) + \delta(a)s^*(a)\left(\frac{\phi_v(0)}{v^*(0)} - \frac{1}{i^*(0)}\right) \\ &+ \beta_s(a)s^*(a)\left(\frac{1}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} - \frac{J^*}{i^*(0)}\right), \end{aligned}$$

employing the equations of $i^*(0)$ and $\phi_v(0)$ we obtain

$$\begin{cases} \dfrac{1}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} - \dfrac{J^*}{i^*(0)} = 0, \\ \dfrac{\phi_v(0)}{v^*(0)} - \dfrac{1}{i^*(0)} = \dfrac{1}{v^*(0)}\dfrac{v^*(0)}{i^*(0)} - \dfrac{1}{i^*(0)} = 0, \end{cases}$$

then

$$L_s(a) = -\frac{1}{i^*(0)}\mu_s(a)s^*(a)$$

Similarly, we prove that :

$$\begin{aligned} L_v(a) &= \phi_v'(a) + \frac{\beta_v(a)v^*(a)}{\int_0^\infty (\beta_s(a)s^*(a) + \beta_v(a)v^*(a))\,da} + \frac{k(a)v^*(a)}{i^*(0)}, \\ &= -\mu_v(a)\frac{v^*(a)}{i^*(0)}. \end{aligned}$$

Finally, the derivative $W'$ verifies the following inequality

$$\begin{aligned} W'(t) &\leq -\frac{1}{i^*(0)}\int_0^\infty H\left(\frac{s(t,a)}{s^*(a)}\right)\mu_s(a)s^*(a)da - \frac{1}{i^*(0)}\int_0^\infty H\left(\frac{v(t,a)}{v^*(a)}\right)\mu_v(a)v^*(a)da, \\ &\leq 0. \end{aligned}$$

We know that $\dfrac{d}{dt}W(t) = 0$ implies that $s(t,a) = s^*(a)$ and $v(t,a) = v^*(a)$ for all $t \in \mathbb{R}$ and $a \geq 0$. We replace these into the first equation of (1), we conclude that $J(t) = J^*$ and so $i(t,0) = i^*(0)$. hence, it follows that $i(t,a) = i^*(a)$ for all $t \in \mathbb{R}$ and $a \geq 0$. Therefore, the largest invariant set with the property $\dfrac{d}{dt}W(t) = 0$ is $\left\{\left(s^*(a), v^*(a), i^*(a)\right)\right\}$. Finally, by employing the same argument as in the proof of Theorem 4.1 in **ST** we reach the result. □

## 7. Discussion

In this study, we investigated the model of imperfect vaccination, obtaining results based on the values of the basic reproduction number, $\mathcal{R}_0$. Specifically, if $\mathcal{R}_0 \leq 1$, we observe the extinction of the disease, as expressed by the stability of the unique equilibrium (trivial equilibrium). Conversely, when $\mathcal{R}_0 > 1$, the disease persists in the population, as indicated by the stability of the second equilibrium (positive equilibrium).

By considering the same set of parameters as in Hathout et al., 2022, the numerical results insure the threshold dynamics obtained in the theoretical part. Indeed, in Fig.2 we remark that the extinction scenario of the infection holds, wherein this case we obtained that $\mathcal{R}_0 = 2.7222.10^{-8} < 1$, which confirms Theorem 6.2. Besides, in Fig.3, we remark the persistence of infection to a positive value, where it is obtained that $\mathcal{R}_0 = 7.0118 > 1$. This figure ensures the main result of Theorem 6.7.

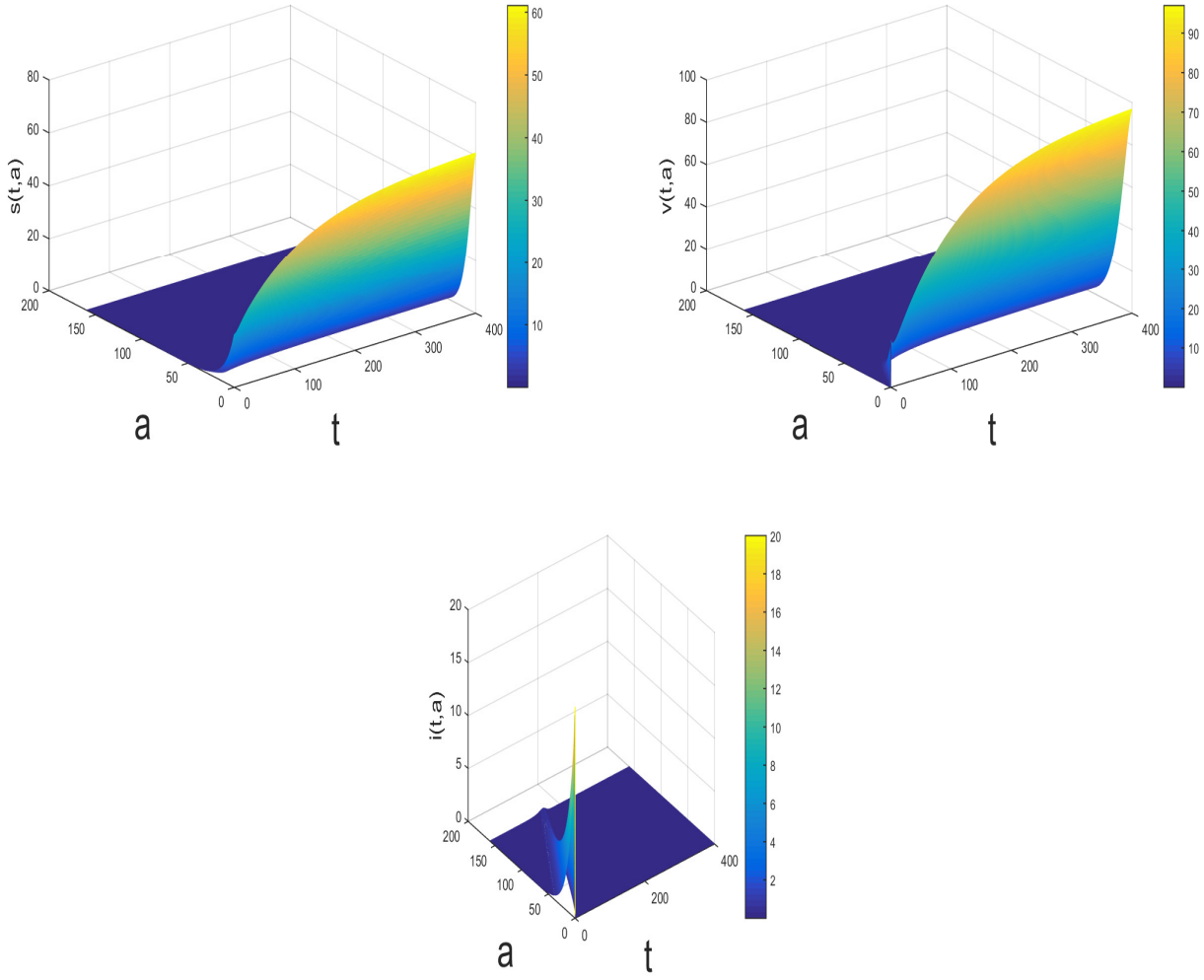


Figure 2: The global stability of the disease free equilibrium in the case of $\mathcal{R}_0 = 2.7222.10^{-8} < 1$.

The issues related to imperfect vaccination are diverse and can pose challenges in the fight against infectious diseases. Some of these problems include the possibility of developing partial immunity after vaccination, the limited duration of immunity, the variable effectiveness of vaccines, selective protection against certain serotypes of pathogens, rare risks of post-vaccination infection, viral adaptation, and variable immune responses in individuals.

Partial immunity may leave some individuals vulnerable to infection, although the severity of the disease may be reduced. Additionally, the waning immunity over time requires regular boosters to maintain adequate protection. The variable effectiveness of vaccines, influenced by factors such as age and general health, can lead to disparities in protection within the population.

Furthermore, the type of vaccination strategy plays a critical role in the success of vaccination efforts. Systematic vaccination campaigns, as seen with diseases like measles, often result in higher coverage and more consistent protection, leading to herd immunity. In contrast, non-systematic vaccination, such as for seasonal flu, may require continuous efforts and adaptation to address seasonal variation and emerging strains, often with lower overall effectiveness in the long run.

To overcome these problems, several solutions can be considered. Firstly, ongoing research to improve the duration of vaccine immunity and develop more durable formulations is crucial. Efforts to understand variable immune responses could allow for vaccine customization based on individual profiles.

Education and communication also play a crucial role. It is important to inform the public about the benefits of vaccination despite its imperfections, emphasizing that even partial protection can reduce the severity

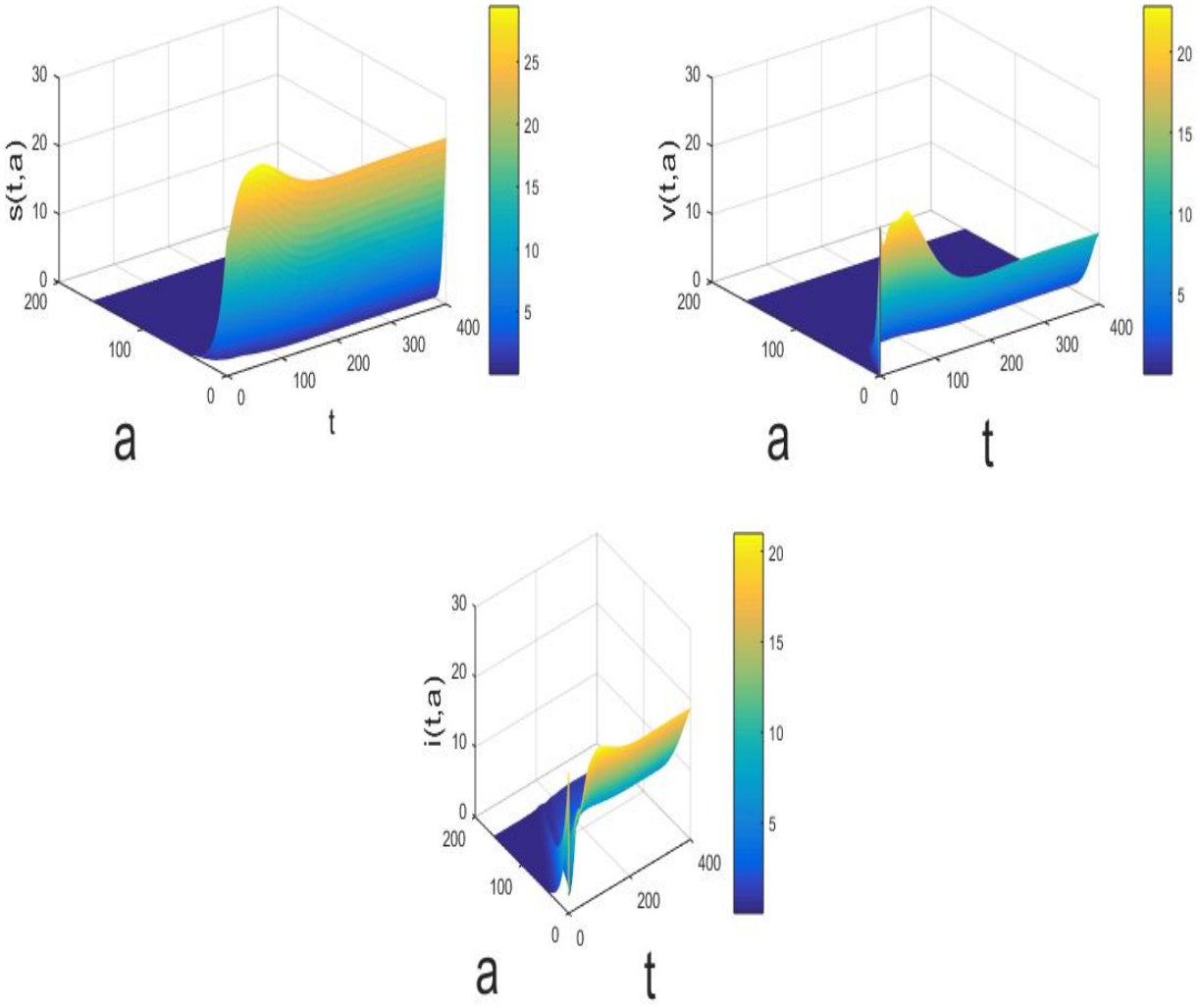


Figure 3: The global stability of the endemic equilibrium in the case of $\mathcal{R}_0 = 7.0118 > 1$.

of diseases and contribute to prevention of spread.

Continuous surveillance of outbreaks and rapid detection of emerging viral strains are fundamental. This could lead to swift adjustments of vaccine formulations to maintain effective protection against new variants.

Ultimately, research, education, surveillance, and constant innovation are key elements in addressing issues associated with imperfect vaccination. By combining these approaches, it is possible to strengthen the fight against infectious diseases and improve the effectiveness of vaccination programs.

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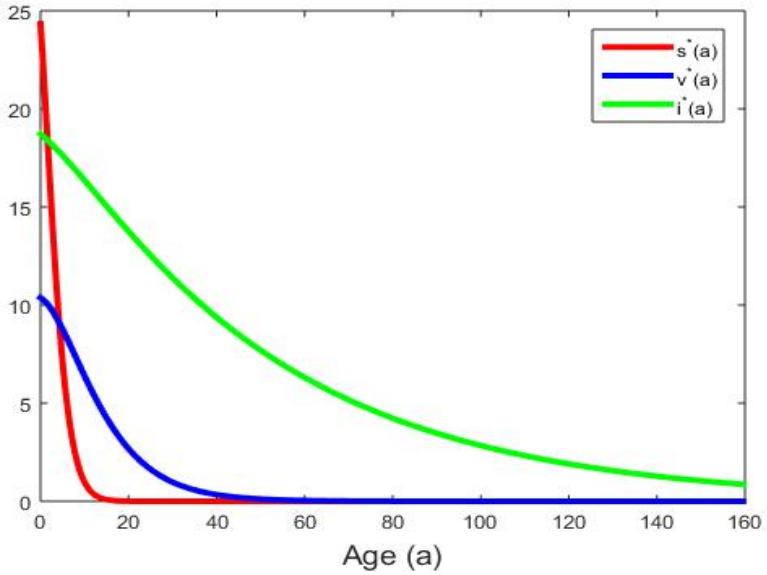


Figure 4: The value of the endemic equilibrium state obtained in Fig.3.

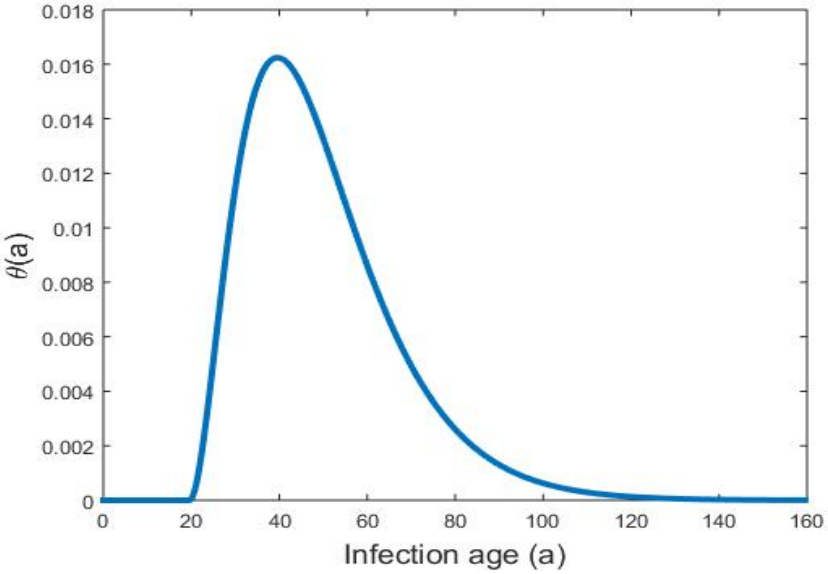


Figure 5: Graphical representation of the infectivity rate $\theta$.

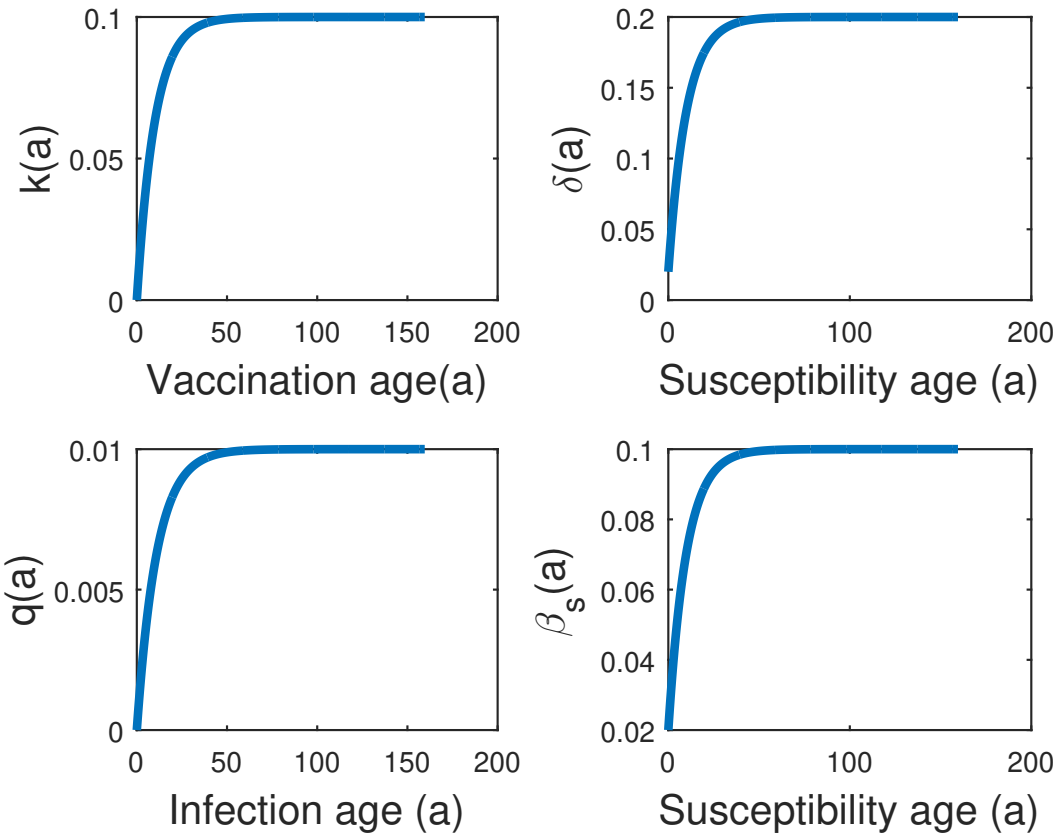


Figure 6: Graphical representation of the functions: $k, \delta, q$ and $\beta_s = \beta_v$.

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