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Erratum: Moderate Deviations Principle and Central Limit Theorem for Stochastic Cahn-Hilliard Equation in Hölder Norm.

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ABSTRACT: We consider a stochastic Cahn-Hilliard partial differential equation driven by a space-time white noise. In this paper, we prove a Central Limit Theorem (CLT) and a Moderate Deviation Principle (MDP) for a perturbed stochastic Cahn-Hilliard equation in Hölder norm. The techniques are based on Freidlin-Wentzell's Large Deviations Principle. The exponential estimates in the space of Hölder continuous functions and the Garsia-Rodemich-Rumsey's lemma plays an important role, an another approach than the Li.R. and Wang.X. Finally, we estabish the CLT and MDP for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients.

Keywords: Large Deviations Principle, Moderate Deviations Principle, Central Limit Theorem, Hölder space, Stochastic Cahn-Hilliard equation, Green's function, Freidlin-Wentzell's method.

MSC: 60H15, 60F05, 35B40, 35Q62

1 Introduction and preliminaries.

The Cahn-Hilliard equation was developed in 1958 to model the phase separation process of a binary mixture (Cahn J.W. and Hilliard J.E. [3,4]). This approach has been extended to many other branches of science as dissimilar as polymer systems, population growth, image processing, spinodal decomposition, among others.

Consider the process $\{X^\varepsilon(t,x)\}_{\varepsilon>0}$ solution of stochastic Cahn-Hilliard with multipicative space time white noise, indexed by $\varepsilon > 0$, given by

$$\begin{cases} \partial_t X^\varepsilon(t,x) = -\Delta(\Delta X^\varepsilon(t,x) - f(X^\varepsilon(t,x))) + \sqrt{\varepsilon}\sigma(X^\varepsilon(t,x))\dot{W}(t,x), \\ \quad \text{in} \ \ (t,x) \in [0,T] \times D, \\ \\ X^\varepsilon(0,x) = X_0(x), \\ \\ \frac{\partial X^\varepsilon(t,x)}{\partial \mu} = \frac{\partial \Delta X^\varepsilon(t,x)}{\partial \mu} = 0, \quad \text{on} \ \ (t,x) \in [0,T] \times \partial D. \end{cases} \tag{1.1}$$

where $T > 0$, $D = [0,\pi]^3$, $\Delta X^\varepsilon(t,x)$ denotes the Laplacian of $X^\varepsilon(t,x)$ in the x-variable, μ is the outward normal vector, f is a polynomial of degree 3 with positive dominant coefficient such as $f = F'$ where

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$F(u) = (1-u^2)^2$, W is a space-time of a Brownian sheet defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is the formal derivative of a Brownian sheet W defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients f, σ are uniform Lipschitz with respect to x, with at most linear growth. More precisely, we suppose that there exists two constants K_f and K_σ such that $\forall x, y \in \mathbb{R}$,

$$\begin{cases} |f(x) - f(y)| \leq K_f |x-y| \\ \\ |\sigma(x) - \sigma(y)| \leq K_\sigma |x-y| \end{cases} \tag{1.2}$$

and that there exists a constant $K > 0$ such that :

$$\sup\{|f(x)|+|\sigma(x)|\} \leq K(1+|x|). \tag{1.3}$$

Let X^0 be the solution of the determinic Cahn-Hilliard equation

$$\partial_t X^0(t,x) = -\Delta(\Delta X^0(t,x) - f(X^0(t,x)))$$

with initial condition $X^0(0,x) = X_0(x)$. We expect that $||X^\varepsilon - X^0||_\alpha \to 0$ in probability as $\varepsilon \to 0^+$ where $||.||_\alpha$ is the Hölder norm (see (2.1)). The LDP, CLT and MDP for stochastic Cahn-Hilliard equation are not new. For example, Boulanba.L. and Mellouk.M. [2] studied the LDP for the mild solution of Stochastic Cahn-Hilliard equation (1.1). Li.R. and Wang.X. [8] studied the CLT and MDP for stochastic perturbed Cahn-Hilliard equation using the weak convergence approach.

However, we study its CLT and MDP for stochastic Cahn-Hilliard equation in the context of Hölder norm using another method. It means, we study the process

$$\eta^\varepsilon(t,x) = \left(\frac{X^\varepsilon - X^0}{\sqrt{\varepsilon}}\right)(t,x) \tag{1.4}$$

and

$$\theta^\varepsilon(t,x) = \left(\frac{X^\varepsilon - X^0}{\sqrt{\varepsilon}h(\varepsilon)}\right)(t,x) \tag{1.5}$$

in order to get a CLT and a MDP respectively.
The techniques are based on the exponential estimates in the space of Hölder continuous functions. The Garsia-Rodemich-Rumsey's lemma plays a very important role.
The paper is organized as follows : in the section one, we prove that $\eta^\varepsilon(t,x)$ defined by (1.4) converges in probability to $\eta^0(t,x)$. More precisely we purpose to prove that $\lim_{\varepsilon\to 0} \mathbb{E}||\eta^\varepsilon - \eta^0||_\alpha^r = 0$. In the section two, we study the LDP for (1.4) as $\varepsilon \to 0$ for $1 < h(\varepsilon) < \frac{1}{\sqrt{\varepsilon}}$, that is to say , the process $\theta^\varepsilon(t,x)$ defined by (1.5) obeys a LDP on $\mathcal{C}^\alpha([0,1]\times D)$ with speed $h^2(\varepsilon)$ and with rate function $\widetilde{I}(.)$ defined later. In section three, we prove the main results. Finally the example for CLT and MDP for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients be given in section four.

2 Main results

Let $\mathbb{H}$ denote the Cameron-Martin space associated with the Brownian sheet $\{W(t,x),\ t\in[0,T],\ \ x \in D\}$, that is to say,

$$\mathbb{H} = \left\{ h(t) = \int_0^t \int_D |\dot{h}(t,x)|^2 dt dx : \dot{h} \in L^2\big([0,T]\times D\big)\right\}.$$

Let $\mathcal{E}_0$, $\mathcal{E}$ be polish space such that the initial condition $X_0(x)$ takes valued in a compact subspace of $\mathcal{E}_0$ and $\Theta^\varepsilon = \{\mathcal{G}^\varepsilon : \mathcal{E}_0 \times \mathcal{C}\big([0,T]\times D, \mathbb{R}\big) \to \mathcal{E},\ \ \varepsilon > 0\}$ a family of measurable maps valued in $\mathcal{E}$.
For $X_0 \in \mathcal{E}_0$, define $X^{\varepsilon, X_0} = \mathcal{G}^\varepsilon\big(X_0, \sqrt{\varepsilon}W\big)$ and for $n_0 \in \mathbb{N}$, consider the following $S^{n_0} = \{\Psi \in L^2([0,T]\times D) : \int_0^T \int_D \Psi^2(s,y) ds dy \leq n_0\}$ which is a compact metric space, equipped with the weak topology on $L^2([0,T]\times D)$.
We denote $||.||_\alpha$ the α-hölder norm such that

$$||F||_\alpha = ||F||_\infty + |F|_\alpha \tag{2.1}$$

where

$$||F||_\infty = \sup\big\{\big|F(s,x)\big| : \ (s,x)\in[0,T]\times D\big\},$$
$$|F|_\alpha = \sup\left\{\frac{|F(s_1,x_1)-F(s_2,x_2)|}{(|s_1-s_2|+|x_1-x_2|^2)^\alpha} : (s_1,x_1),(s_2,x_2)\in[0,T]\times D\right\}.$$

Let $\mathcal{C}^\alpha([0,T]\times D)$ the space of function $F:[0,T]\times D\longrightarrow\mathbb{R}$ such that $||F||_\alpha<+\infty$.
Schilder's theorem for the Brownian sheet asserts that the family
$\big\{\sqrt{\varepsilon}W(t,x):\ \varepsilon>0\big\}$ satisfies a LDP on $\mathcal{C}^\alpha([0,T]\times D)$, with the good rate function $I(.)$ defined by

$$I(h)=\begin{cases}\frac{1}{2}\int_0^T\int_D|\dot{h}(t,x)|^2dtdx & \text{for } h\in\mathbb{H}\\ +\infty & \text{otherwise,}\end{cases}$$

For $h\in\mathbb{H}$, let $X^h_{X_0}$ be the solution of the following deterministic partial differential equation

$$\partial_t X^h_{X_0}(t,x)=-\Delta(\Delta X^h_{X_0}(t,x)-f(X^h_{X_0}(t,x)))+\sigma(X^h_{X_0}(t,x))\dot{h}(t,x)$$

with initial condition

$$X^h_{X_0}(0,x)=X_0(x).$$

Theorem 1([2]): *Let σ be continuous on $\mathbb{R}$, f and σ satisfy conditions (1.2) and (1.3). Then, the law of $X^\varepsilon_{X_0}$ satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times D)$ with a good rate fuction $\widetilde{I}_{X_0}(.)$ defined by*

$$\widetilde{I}_{X_0}(\Phi)=\inf_{\left\{\dot{h}\in L^2([0,T]\times D)\,:\,\Phi=\mathcal{G}^0(X_0,I(h))\right\}}\left\{\frac{1}{2}\int_0^T\int_D\dot{h}^2(s,y)dsdy\right\}$$

and $+\infty$ otherwise.
See also for example [1,7].
In addition to (1.2) and (1.3), the coefficient f is differentiable with respect to x and the derivative $f^{'}$ is also uniformly Lipschitz. More precisely, there exists a constante C such that

$$|f^{'}(x)-f^{'}(y)|\le C|x-y| \tag{2.2}$$

for all $x,y\in\mathbb{R}$.
Combined with the uniform Lipschitz continuity of f, we have

$$|f^{'}(x)|\le K_f. \tag{2.3}$$

2.1 Central Limit Theorem

In this section, our first main result is the following theorem :

Theorem 2: *Suppose that f, $f^{'}$ and σ satisfy conditions (1.2), (1.3), (2.2) and (2.3). Then for any $\alpha\in[0;\frac{1}{4})$, $r\ge1$, the process $\eta^\varepsilon(t,x)$ defined by (1.4) converges in L^r to the random process $\eta^0(t,x)$ as $\varepsilon\to0$ where $\eta^0(t,x)$ verifies the stochastic partial differential equation*

$$\partial_t\eta^0(t,x)=-\Delta(\Delta\eta^0(t,x)-f^{'}(X^0(t,x))\eta^0(t,x))+\sigma(X^0(t,x))\dot{W}(t,x)$$

with initial condition $\eta^0(0,x)=0$.

Let $S(t)=e^{-A^2t}$ be the semi-group generated by the operator $A^2u:=\sum_{i=0}^\infty e^{-\mu_i^2t}u_iw_i$ where $u:=\sum_{i=0}^\infty u_iw_i$. Then the convolution semi-group (see Cardon-Weber.C [5]) is defined by $S(t)U(x)=\sum_{i=0}^\infty e^{-\mu_i^2t}w_i(x)w_i(y)$ for any $U(x)$ in $L^2(D)$, with the associated Green's function G_t such that $G_t(x,y)=\sum_{i=0}^\infty e^{-\mu_i^2t}w_i(x)w_i(y)$. **Lemma 1:** *There exists positive constants C, γ and $\gamma^{'}$ satisfying $\gamma<4-d$, $\gamma\le2$ and $\gamma^{'}<1-\frac{d}{4}$ such that for all $y,z\in D$, $0\le s<t\le T$ and $0\le h\le t$, we have :*

1. $\int_0^t\int_D|G_r(x,y)-G_r(x,z)|^2dxdr\le C|y-z|^\gamma$,
2. $\int_0^t\int_D|G_{r+h}(x,y)-G_r(x,y)|^2dxdr\le C|h|^{\gamma^{'}}$,

3. $\int_0^t \int_D |G_r(x,y)|^2 dxdr \leq C|t-s|^\gamma$,
4. $\sup_{t\in[0,T]} \int_0^t \int_D |G_{t-u}(x,z) - G_{t-u}(y,z)|^p dudz \leq C|x-y|^{3-p}$, $p \in]\frac{3}{2}, 3[$,
5. $\sup_{x\in D} \int_0^s \int_D |G_{t-u}(x,z) - G_{s-u}(x,z)|^p dudz \leq C|t-s|^{\frac{(3-p)}{2}}$, $p \in]1,3[$,
6. $\sup_{x\in D} \int_t^s \int_D |G_u(x,z)|^p dudz \leq C|t-s|^{\frac{(3-p)}{2}}$, $p \in]1,3[$.

2.2 Moderate Deviations Principle

In this paper, our second main result is the MDP for the Stochastic Cahn-Hilliard equation. More precisely, we assume that the process $\{\theta^\varepsilon(t,x)\}_{\varepsilon>0}$ defined by (1.5) obeys a LDP on the space $\mathcal{C}^\alpha([0,1]\times D)$, with speed $h^2(\varepsilon)$ and rate function $\widetilde{I}_{X_0}(.)$.

Proposition 1: *If f and σ are Lipschitzian, then there exists $C(p,K,K_f,$ $T,X_0)$ depending on p, K, K_f, T, X_0 such that*

$$\mathbb{E}\big(||X^\varepsilon - X^0||_\infty\big)^p \leq \varepsilon^{\frac{p}{2}} C(p,K,K_f,T,X_0) \longrightarrow 0 \ \text{as} \ \varepsilon \to 0.$$

Theorem 3: *Let σ be continuous on $\mathbb{R}$ and f, f', σ satisfy the conditions (1.2), (1.3), (2.2) and (2.3).Then, the process $\{\theta^\varepsilon(t,x)\}_{\varepsilon>0}$ defined by (1.5) obeys a LDP on the space $\mathcal{C}^\alpha([0,1]\times D)$, with speed $h^2(\varepsilon)$ and rate function $\widetilde{I}_{X_0}(.)$ such that:*

$$\widetilde{I}_{X_0}(\phi) = \inf_{\{\dot{h}\in L^2([0,T]\times D)\,:\,\phi=\mathcal{G}^0(X_0,I(h))\}} \left\{ \frac{1}{2}\int_0^T \int_D \dot{h}^2(s,y)dyds \right\}$$

and $+\infty$ otherwise.

3 Proof of main results

Proof of proposition 1: In Boulanba and Mellouk [2], we know that the stochastic Cahn-Hilliard equation has a solution $\{X^\varepsilon(t,x)\}_{\varepsilon>0}$ such that

$$\begin{aligned} X^\varepsilon(t,x) &= \int_D G_t(x,y)X_0(y)dy + \int_0^t \int_D \Delta G_{t-s}(x,y) f(X^\varepsilon(s,y))dsdy \\ &+ \sqrt{\varepsilon}\int_0^t \int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy). \end{aligned}$$

and that $||X^\varepsilon - X^0||_\alpha \to 0$ in probability as $\varepsilon \to 0^+$ where X^0 is the solution of

$$X^0(t,x) = \int_D G_t(x,y)X_0(y)dy + \int_0^t \int_D \Delta G_{t-s}(x,y) f(X^0(s,y))dsdy.$$

Then we have

$$\begin{aligned} \big(X^\varepsilon - X^0\big)(t,x) &= \int_0^t \int_D \Delta G_{t-s}(x,y)\big[f(X^\varepsilon(s,y)) - f(X^0(s,y))\big]dsdy \\ &+ \sqrt{\varepsilon}\int_0^t \int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy). \end{aligned}$$

Using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, we have

$$\begin{aligned} \big(||X^\varepsilon - X^0||_\infty\big)^p &\leq 2^{p-1}\Bigg(\Bigg[\sup_{\substack{0\leq s\leq T\\ x\in D}}\Bigg|\int_0^t \int_D \Delta G_{t-s}(x,y)[f(X^\varepsilon(s,y)) \\ &- f(X^0(s,y))]dsdy\Bigg|\Bigg]^p \\ &+ \varepsilon^{\frac{p}{2}}\Bigg[\sup_{\substack{0\leq s\leq T\\ x\in D}}\Bigg|\int_0^t \int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy)\Bigg|\Bigg]^p\Bigg). \end{aligned}$$

Denote

$$\begin{aligned}\alpha_1^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)[f(X^\varepsilon(s,y))-f(X^0(s,y))]dsdy,\\ \alpha_2^\varepsilon(t,x) &= \int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

From (1.2), (1.3) and Hölder inequality, for $p>2$,

$$\mathbb{E}\big(||\alpha_1^\varepsilon||_\infty^T\big)^p \le K_f^p\Big(\sup_{\substack{0\le s\le T\\ x\in D}}\Big|\int_0^t\int_D \Delta G_t^q(x,y)dsdy\Big|\Big)^{\frac{p}{q}}\mathbb{E}\int_0^T |X_{X_0}^\varepsilon - X_{X_0}^0|^p dt$$

where $\frac{1}{p}+\frac{1}{q}=1$.

For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$ such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, and for any $x,y\in D$, $t\in[0,T]$, by Burkholder's inequality for stochastic integrals against Brownian sheets (see Walsh.J.B. [9], page 315) and Hölder's inequality, we have

$$\begin{aligned}&\mathbb{E}\big(|\alpha_2^\varepsilon(t,x)-\alpha_2^\varepsilon(t,y)|^p\big)\\ &\le c_p\mathbb{E}\bigg(\int_0^t\int_D |G_{t-u}(x,z)-G_{t-u}(y,z)|^2\sigma^2(X_{X_0}^\varepsilon(u,z))dudz\bigg)^{\frac{p}{2}}\\ &\le c_pK^p\bigg(\int_0^t\int_D |G_{t-u}(x,z)-G_{t-u}(y,z)|^{2q'}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\bigg(\int_0^t\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\bigg)^{\frac{p}{2p^{'}}}\\ &\le C(p,K,X_0)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}},\end{aligned}\tag{3.1}$$

where (1.3) and 4 in Lemma 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$ and $C(p,K,X_0)$ is independent of ε.

Similary, from 4, 5 and 6 in Lemma 1, for $0\le s\le t\le T$,

$$\begin{aligned}&\mathbb{E}\big(|\alpha_2^\varepsilon(t,y)-\alpha_2^\varepsilon(s,y)|^p\big)\\ &\le c_p\mathbb{E}\bigg(\int_0^s\int_D |G_{t-u}(y,z)-G_{s-u}(y,z)|^2\sigma^2(X_{X_0}^\varepsilon(u,z))dudz\bigg)^{\frac{p}{2}}\\ &\quad+c_p\mathbb{E}\bigg(\int_s^t\int_D |G_{t-u}(y,z)|^2\sigma^2(X_{X_0}^\varepsilon(u,z))dudz\bigg)^{\frac{p}{2}}\\ &\le c_pK^p\bigg(\int_0^s\int_D |G_{t-u}(y,z)-G_{s-u}(y,z)|^{2q'}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\bigg(\int_0^s\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\bigg)^{\frac{p}{2p^{'}}}\\ &\quad+c_pK^p\bigg(\int_s^t\int_D |G_{t-u}(y,z)|^{2q'}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\quad\times\mathbb{E}\bigg(\int_s^t\int_D (1+|X_{X_0}^\varepsilon(u,z)|)^{2p'}dudz\bigg)^{\frac{p}{2p^{'}}}\\ &\le C(p,K,X_0)|t-s|^{\frac{(3-2q^{'})p}{4q^{'}}}\end{aligned}\tag{3.2}$$

Putting together (3.1) and (3.2), by Garsia-Rodemich-Rumsey (see Wang.R. and Zang.T. [10] or Corollary 1.2 in Walsh.J.B. [9]), there exist a random variable $K_{p,\varepsilon}(\omega)$ and a constant c such that

$$\begin{aligned}
&\mathbb{E}\big(|\alpha_2^\varepsilon(t,y)-\alpha_2^\varepsilon(s,y)|^p\big)\\
&\quad\leq\ K_{p,\varepsilon}(\omega)^p(|t-s|+|x-y|)^\gamma\left(\log\frac{c}{|t-s|+|x-y|}\right)^2 \qquad (3.3)
\end{aligned}$$

and

$$\sup_{\varepsilon}\mathbb{E}[K^p_{p,\varepsilon}]<+\infty.$$

choosing $s=0$ in (3.3), we obtain

$$\begin{aligned}
\mathbb{E}\big(\sup_{\substack{0\leq s\leq T\\ x\in D}}\big|\int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy)\big|\big)^p &\leq C(p,K,X_0)\sup_{\varepsilon}\mathbb{E}[K^p_{p,\varepsilon}]\\
&< +\infty. \qquad (3.4)
\end{aligned}$$

Putting (3.1), (3.2) and (3.3) together and using 6 in Lemma 1, there exists a constant $C(p,K,K_f,X_0)$ such that

$$\mathbb{E}(||X_t^\varepsilon-X_t^0||_\infty^T)^p \leq C(p,K,K_f,X_0)\left(\mathbb{E}\int_0^t(||X_s^\varepsilon-X_s^0||_\infty)^p ds+\varepsilon^{\frac{p}{2}}\right)$$

By Gronwall's inequality, we have

$$\mathbb{E}(||X_t^\varepsilon-X_t^0||_\infty)^p \leq \varepsilon^{\frac{p}{2}}C(p,K,K_f,X_0)e^{C(p,K,K_f,X_0)T}.$$

Putting $\varepsilon\to 0$, the proof is complete. □

Proof of Theorem 2 : The following Lemma is a consequence of Garsia-Rodemich-Rumsey's theorem.

Lemma 2: *Let $\widetilde{V}^\varepsilon(t,x)=\{V^\varepsilon(t,x)\ :\ (t,x)\in[0,T]\times D\}$ be a family of real-valued stochastic processes and let $p\in(0,\infty)$. Suppose that $\widetilde{V}^\varepsilon(t,x)$ satisfies the following assumptions :*

A-1°) *For any $(t,x)\in[0,T]\times D$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}|V^\varepsilon(t,x)|^p=0$$

A-2°) *There exists $\gamma>0$ such that for any (t,x), $(s,y)\in[0,T]\times D$*

$$\mathbb{E}|V^\varepsilon(t,x)-V^\varepsilon(s,y)|^p\leq C(|t-s|+|x-y|^2)^{2+\gamma},$$

where C is a constant independent of ε.
In this case, for any $\alpha\in(0,\frac{\gamma}{k})$, $p\in[1,k)$,

$$\lim_{\varepsilon\to 0}\mathbb{E}||V^\varepsilon||_\alpha^p=0.$$

In this section, we prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}||X_t^\varepsilon-X_t^0||_\alpha^r=0.$$

Consider the process $\eta^\varepsilon(t,x)$ defined by (1.4) and

$$\begin{aligned}
X^\varepsilon(t,x) &= \int_D G_t(x,y)X_0(y)dy+\int_0^t\int_D\Delta G_{t-s}(x,y)f(X^\varepsilon(s,y))dsdy\\
&+ \sqrt{\varepsilon}\int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy).
\end{aligned}$$

We know that $||X^\varepsilon-X^0||_\alpha\to 0$ in probability as $\varepsilon\to 0^+$ where X^0 is the solution of

$$X^0(t,x) = \int_D G_t(x,y)X_0(y)dy+\int_0^t\int_D\Delta G_{t-s}(x,y)f(X^0(s,y))dsdy.$$

In this case, we have

$$\begin{aligned}\eta^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^\varepsilon(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\Big)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy)\end{aligned}$$

then

$$\begin{aligned}\eta^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^\varepsilon(s,y))\eta^\varepsilon(s,y)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

For $\varepsilon\to 0$, we have

$$\begin{aligned}\eta^0(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^0(s,y))\eta^0(s,y)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\sigma(X^0(s,y))W(ds,dy).\end{aligned}$$

To this end, we verify (A-1), (A-2); for $V^\varepsilon=\eta^\varepsilon-\eta^0$, write

$$\begin{aligned}V^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^\varepsilon(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\\ &- f^{'}(X^0(s,y))\eta^0(s,y)\Big)dsdy\\ &+ \int_0^t\int_D G_{t-s}(x,y)\big(\sigma(X^\varepsilon(s,y))-\sigma(X^0(s,y))\big)W(ds,dy).\end{aligned}$$

Let

$$\begin{aligned}k_1^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)\Big(\frac{f(X^\varepsilon(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}\\ &\quad -f^{'}(X^0(s,y))\eta^\varepsilon(s,y)\Big)dsdy,\\ k_2^\varepsilon(t,x) &= \int_0^t\int_D \Delta G_{t-s}(x,y)f^{'}(X^0(s,y))\big(\eta^\varepsilon(s,y)-\eta^0(s,y)\big)dsdy,\\ k_3^\varepsilon(t,x) &= \int_0^t\int_D G_{t-s}(x,y)\big(\sigma(X^\varepsilon(s,y))-\sigma(X^0(s,y))\big)W(ds,dy).\end{aligned}$$

Now we shall divide the proof into the following two steps.

Step 1. Following the same calculation as the proof of (3.4) in proposition 1, we deduce that for $p>2$, $0\le t\le 1$

$$\begin{aligned}\mathbb{E}\big(||k_3^\varepsilon||_\infty^t\big) &\le C(p,K_\sigma,T)\int_0^t \mathbb{E}\big(||X^\varepsilon-X^0||_\infty^s\big)^p ds\\ &\le \varepsilon^{\frac{p}{2}}C(p,K,K_\sigma,T,X_0).\end{aligned}$$

By Taylor's formula, there exists a random field $\beta^\varepsilon(t,x)$ taking values in $(0,1)$ such that,

$$\begin{aligned}f(X^\varepsilon(s,y))-f(X^0(s,y)) &= f^{'}\big(X^0(s,y)+\beta^\varepsilon(t,x)(X^\varepsilon(s,y)-X^0(s,y))\big)\\ &\quad\times(X^\varepsilon(s,y)-X^0(s,y))\end{aligned}$$

Since $f^{'}$ is also Lipschitz continuous, we have

$$\big|f^{'}\big(X^0(s,y)+\beta^\varepsilon(t,x)(X^\varepsilon(s,y)-X^0(s,y))\big)-f^{'}\big(X^0(s,y)\big)\big|$$

$$\leq C\beta^{\varepsilon}(t,x)\big|X^{\varepsilon}(t,x)-X^{0}(t,x)\big|.$$

then

$$\big|f^{'}\big(X^{0}(s,y)+\beta^{\varepsilon}(t,x)(X^{\varepsilon}(s,y)-X^{0}(s,y))\big)-f^{'}\big(X^{0}(s,y)\big)\big| \leq C\big|X^{\varepsilon}(t,x)-X^{0}(t,x)\big|.$$

Hence

$$\begin{aligned}\big|k_1^{\varepsilon}(t,x)\big| &\leq C\int_0^t\int_D \Delta G_{t-s}(x,y)\big|\big(X^{\varepsilon}(t,x)-X^{0}(t,x)\big)\eta^{\varepsilon}(s,y)\big|dsdy\\ &= \sqrt{\varepsilon}C\int_0^t\int_D \Delta G_{t-s}(x,y)\big(\eta^{\varepsilon}(s,y)\big)^2dsdy. \end{aligned}\tag{3.5}$$

By Hölder's inequality, for $p>2$

$$\begin{aligned}&\mathbb{E}\big(\big|k_1^{\varepsilon}\big|_{\infty}^{t}\big)^p\\ &\leq \varepsilon^{\frac{p}{2}}C^p\bigg(\sup_{0\leq s\leq T\ \ ,x\in D}\ \bigg|\int_0^t\int_D \Delta G_s^q(x,y)dsdy\bigg|\bigg)^{\frac{p}{q}}\times\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}||_{\infty}^{s}\big)^{2p}ds\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Using (2.2) and applying proposition 1, there exists a constant $C(p,K,K_f,C,K_\sigma,T,X_0)$ depending on p, K, K_f, C, K_σ, T, X_0 such that

$$\mathbb{E}\big(\big|k_1^{\varepsilon}(t,x)\big|\big)^p \leq \varepsilon^{\frac{1}{2}}C(p,K,K_f,C,K_\sigma,T,X_0)\tag{3.6}$$

Noticing that $|f^{'}|\leq K_f$, by Hölder inequality, we deduce that for $p>2$

$$\begin{aligned}&\mathbb{E}\big(\big|k_2^{\varepsilon}(t,x)\big|\big)^p\\ \leq\ & K_f^p\big(\sup_{\substack{0\leq s\leq T\\ x\in D}}\ \big|\int_0^t\int_D \Delta G_s^q(x,y)dsdy\big|\big)^{\frac{p}{q}}\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p ds\end{aligned}\tag{3.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Putting (3.5), (3.6) and (3.7) together, we have

$$\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p\leq C(p,K,K_f,C,K_\sigma,T,X_0)\bigg(\varepsilon^{\frac{1}{2}}+\int_0^t\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p ds\bigg)$$

By Gronwall's inequality, we obtain

$$\mathbb{E}\big(||\eta^{\varepsilon}-\eta^0||_{\infty}^{s}\big)^p\leq \varepsilon^{\frac{1}{2}}C(p,K,K_b,C,K_\sigma,T,X_0)\longrightarrow 0\ \text{ for }\ \varepsilon\to 0.$$

Step 2. We show that all the terms k_i^ε, $i=1,2,3$ satisfy the condition (A-2) in Lemma 2. For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$ such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, for all $x,y\in D$, $0\le t\le T$, by Burkholder's inequality and Hölder's inequality, we have

$$\begin{aligned}\mathbb{E}\big|k_3^\varepsilon(t,x)-k_3^\varepsilon(t,y)\big|^p &\le C_p\mathbb{E}\bigg(\int_0^t\int_D|G_{t-u}(x,z)-G_{t-u}(y,z)|^2\\ &\qquad\times(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z)))^2dudz\bigg)^{\frac{p}{2}}\\ &\le C_p\bigg(\int_0^t\int_D(|G_{t-u}(x,z)-G_{t-u}(y,z)|)^{2q^{'}}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\qquad\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ &\le C(p,q^{'},K_\sigma,K,T)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}}\end{aligned}\tag{3.8}$$

where (1.3), 4 in Lemma 1 and Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$.
Similarly, in view of 5 , 6 in Lemma 1; it follows that for $0\le s\le t\le T$, we have

$$\begin{aligned}&\mathbb{E}\big|k_3^\varepsilon(t,y)-k_3^\varepsilon(s,y)\big|^p\\ \le\;& C_p\mathbb{E}\Big(\int_0^s\int_D|G_{t-u}(y,z)-G_{s-u}(y,z)|^2\big(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z))\big)^2dudz\Big)^{\frac{p}{2}}\\ +\;& C_p\mathbb{E}\Big(\int_s^t\int_D|G_{t-u}(y,z)|^2\big(\sigma(X^\varepsilon(u,z))-\sigma(X^0(u,z))\big)^2dudz\Big)^{\frac{p}{2}}\\ \le\;& C_p\bigg(\int_0^t\int_D|G_{t-u}(y,z)-G_{s-u}(y,z)|^{2q^{'}}dudz\bigg)^{\frac{p}{2q^{'}}}\\ &\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ +\;& C_p\Big(\int_s^t\int_D|G_{t-u}(y,z)|^{2q^{'}}dudz\Big)^{\frac{p}{2q^{'}}}\\ &\times K_\sigma^p\mathbb{E}\bigg(\int_0^t\int_D|X^\varepsilon(u,z)-X^0(u,z)|^{2p^{'}}dudz\bigg)^{\frac{p}{2p^{'}}}\\ \le\;& C(p,q^{'},K_\sigma,K,T)|t-s|^{\frac{(3-2q^{'})p}{4q^{'}}}\end{aligned}\tag{3.9}$$

where Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$, $C(p,q^{'},K_\sigma,K,T)$ is independent of ε.
Putting together (3.8) and (3.9), we have

$$\mathbb{E}\big|k_3^\varepsilon(t,x)-k_3^\varepsilon(s,y)\big|^p\le C(p,q^{'},K_\sigma,K,T)\big(|t-s|+|x-y|^2\big)^\gamma\tag{3.10}$$

Consequently, from 4, 6 in Lemma 1, proposition 1 and the result of step 1, we also have :

$$\mathbb{E}\big|k_i^\varepsilon(t,x)-k_i^\varepsilon(s,y)\big|^p\le C\big(|t-s|+|x-y|^2\big)^\gamma\ ,\ \ i=2,3.\tag{3.11}$$

Putting together (3.10) and (3.11), we obtain that there exists a constant C independent of ε satisfying that

$$\mathbb{E}\big|(\eta^\varepsilon(t,x)-\eta^0(t,x))-(\eta^\varepsilon(s,y)-\eta^0(s,y))|^p\le C\big(|t-s|+|x-y|^2\big)^\gamma$$

For any $\alpha\in(0,\frac{1}{4})$, $r\ge1$, choosing $p>2$, and $q^{'}\in(1,\frac{1}{4})$ such that $\alpha\in(0,\frac{\gamma}{p})$ and $r\in[1,p)$, Lemma 2 we have

$$\lim_{\varepsilon\to0}\mathbb{E}||\eta^\varepsilon-\eta||_\alpha^r=0.$$

The proof is complete . □

Proof of Theorem 3 : Recall the following lemma from Chenal.F and Millet.A [6].

Lemma 3: *Let $F : ([0,T]\times D)^2 \longrightarrow \mathbb{R}$, $\alpha_0 > 0$ and $C_F > 0$ be such that for any $(t,x),(s,y) \in [0,T]\times D$, set*

$$\int_0^T \int_D |F(t,x,u,z) - F(s,y,u,z)|^2 dudz \leq C(|t-s| + |x-y|^2)^{\alpha_0}. \tag{3.12}$$

Let $N : [0,T]\times D \longrightarrow \mathbb{R}$ be an almost surely continuous, $\mathcal{F}_t$−adapted such that $\sup\{|N(t,x)| : (t,x) \in [0,T]\times D\} \leq \rho$,a.s., and for $(t,x) \in [0,T]\times D$, set

$$\mathfrak{F}(t,x) = \int_0^T \int_D F(t,x,u,z)N(u,z)W(dudz)$$

Then for all $\alpha \in]0, \frac{\alpha_0}{2}[$, there exists a constant $C(\alpha,\alpha_0)$ such that for all $M \geq \rho C_F C(\alpha,\alpha_0)$

$$\mathbb{P}(||\mathfrak{F}||_\alpha \geq M) \leq (\sqrt{2}T^2 + 1)\exp\left(- \frac{M^2}{\rho^2 C_F C^2(\alpha,\alpha_0)}\right)$$

Proof of Theorem 3 : Now, we prove the MDP, that is to say, the process θ^ε defined by (1.5) obeys a LDP on $\mathcal{C}^\alpha([0,T]\times D)$, with the speed function $h^2(\varepsilon)$ and the rate function $\widetilde{I}(.)$. More precisely, to prove the LDP of $\frac{\eta^\varepsilon}{h(\varepsilon)}$, it is enough to show that $\frac{\eta^\varepsilon}{h(\varepsilon)}$ is $h^2(\varepsilon)$-exponentially equivalent to $\frac{\eta^0}{h(\varepsilon)}$,that is to say, for any $\delta > 0$, we have

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\left(\frac{||\eta^\varepsilon - \eta^0||_\alpha}{h(\varepsilon)} > \delta\right) = -\infty. \tag{3.13}$$

Since

$$||\eta^\varepsilon - \eta^0||_\alpha \leq (1 + (1+T)^\alpha)|\eta^\varepsilon - \eta^0|_\alpha^T$$

to prove (3.13), it is enough to prove that

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\left(\frac{|\eta^\varepsilon - \eta^0|_\alpha^T}{h(\varepsilon)} > \delta\right) = -\infty \quad , \qquad \forall\delta > 0.$$

Recall the decomposition in Proof of Theorem 2,

$$\eta^\varepsilon(t,x) - \eta^0(t,x) = k_1^\varepsilon(t,x) + k_2^\varepsilon(t,x) + k_3^\varepsilon(t,x).$$

For any q in $(\frac{3}{2},3)$, $\frac{1}{p}+\frac{1}{q} = 1$, and $x,y \in D$, $0 \leq s \leq t \leq T$, by Hölder's inequality, 4 in Lemma 1 and (2.3), we have

$$\begin{aligned}
\left|k_2^\varepsilon(t,x) - k_2^\varepsilon(t,y)\right|^p &\leq K_f\left(\int_0^t \int_D |\Delta G_{t-u}(x,z) - \Delta G_{t-u}(y,z)|^q dudz\right)^{\frac{1}{q}} \\
&\quad \times\left(\int_0^t \int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p dudz\right)^{\frac{1}{p}} \\
&\leq K_f|x-y|^{\frac{3-q}{q}} \times \left(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\right)^{\frac{1}{p}}
\end{aligned} \tag{3.14}$$

Similarly, in view of 5 and 6 in Lemma 1, it follows that for $0 \leq s \leq t \leq T$,

$$\begin{aligned}\big|k_2^\varepsilon(t,y) - k_2^\varepsilon(s,y)\big|^p &\leq K_f\bigg(\int_0^s\int_D |\Delta G_{t-u}(y,z) - \Delta G_{s-u}(y,z)|^q dudz\bigg)^{\frac{1}{q}} \\ &\quad \times\bigg(\int_0^s\int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p\bigg)^{\frac{1}{p}} \\ &\quad +\bigg(\int_s^t\int_D |\Delta G_{t-u}(y,z)|^q dudz\bigg)^{\frac{1}{q}} \\ &\quad \times\bigg(\int_0^t\int_D |\eta^\varepsilon(u,z) - \eta^0(u,z)|^p\bigg)^{\frac{1}{p}} \\ &\leq 2K_f|t-s|^{\frac{3-q}{2q}} \times \bigg(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\bigg)^{\frac{1}{p}} \end{aligned} \tag{3.15}$$

Putting together (3.14), (3.15), we have

$$\big|k_2^\varepsilon(t,y) - k_2^\varepsilon(s,y)\big|^p \leq C(K_f)(|t-s| + |x-y|^2)^{\frac{3-q}{2q}} \times \bigg(\int_0^t (||\eta^\varepsilon - \eta^0||_\infty^u)^p du\bigg)^{\frac{1}{p}}.$$

Choosing $q \in (\frac{3}{2}, 3)$, such that $\alpha = (3-q)/2q$ and noticing that $||\eta^\varepsilon - \eta^0||_\infty^u \leq (1+u)^\alpha|\eta^\varepsilon - \eta|_\alpha^u$, we obtain that

$$|k_2^\varepsilon|_\alpha^t \leq C(K_f)\bigg(\int_0^t ((1+u)^\alpha|\eta^\varepsilon - \eta^0|_\alpha^u)^p du\bigg)^{\frac{1}{p}}$$

Thus, for $t \in [0,1]$, we have

$$(|\eta_t^\varepsilon - \eta_t^0|_\alpha^t)^p \leq C(p,T,K_f)\bigg[\big(|k_1^\varepsilon(t)|_\alpha^t + |k_3^\varepsilon(t)|_\alpha^t\big)^p + \int_0^t (|\eta^\varepsilon - \eta^0|_\alpha^s)^p ds\bigg]$$

Applying Gronwall's Lemma, we have

$$(|\eta_t^\varepsilon - \eta_t^0|_\alpha^t)^p \leq C(p,T,K_f)\bigg[\big(|k_1^\varepsilon(t)|_\alpha^t + |k_3^\varepsilon(t)|_\alpha^t\big)^p\bigg] e^{C(p,T,K_f)T} \tag{3.16}$$

By (3.15) and (3.16), its sufficient to prove that for any $\delta > 0$

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{|k_i^\varepsilon(t)|_\alpha^T}{h(\varepsilon)} > \delta\bigg) = -\infty \qquad\qquad i = 1,3.$$

Step 1. For any $\varepsilon > 0$, $\eta > 0$ we have

$$\begin{aligned}\mathbb{P}\big(|k_3^\varepsilon|_\alpha^T > h(\varepsilon)\delta\big) &\leq \mathbb{P}\big(|k_3^\varepsilon|_\alpha^T > h(\varepsilon)\delta, |X^\varepsilon - X^0|_\infty^T < \eta\big) \\ &+ \mathbb{P}(|X^\varepsilon - X^0|_\infty^T \geq \eta)\end{aligned} \tag{3.17}$$

By 4 and 6 in Lemma 1, $G_{t-u}(x,z)1_{[u\leq t]}$ satisfies (3.12)(see Lemma 3) for $\alpha_0 = \frac{1}{2}$. Applying Lemma 3, we have

$$F(t,x,u,z) = G_{t-u}(x,z)1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, M = h(\varepsilon)\delta, \rho = \eta K_\sigma,$$

$$\widetilde{Y}(t,x) = \big(\sigma(X_{X_0}^\varepsilon(t,x)) - \sigma(X_{X_0}^0(t,x))\big)1_{||X^\varepsilon - X^0||_\infty^T > \eta}$$

, we obtain that for all ε sufficiently small such that $h(\varepsilon)\delta \geq \rho CC(\alpha, \frac{1}{2})$,

$$\begin{aligned}&\mathbb{P}\big(|k_3^\varepsilon(t)|_\alpha^T > h(\varepsilon)\delta, ||X^\varepsilon - X^0||_\infty^T < \eta\big) \\ &\quad \leq (\sqrt{2}T^2 + 1)\exp\bigg(-\frac{h^2(\varepsilon)\delta^2}{\eta^2 K_\sigma^2 CC^2(\alpha, \frac{1}{2})}\bigg).\end{aligned} \tag{3.18}$$

Since $X^\varepsilon_{X_0}$ satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times D)$, see Theorem 1

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(||X^\varepsilon-X^0||^T_\infty\geq\eta) &\leq \limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(||X^\varepsilon-X^0||_\alpha\geq\eta)\\ &\leq -\inf\{I_{X_0}(f):\ ||f-X^0||_\alpha\geq\eta\}\end{aligned}$$

In this case, the good rate function $\mathcal{I}=\{I_{X_0}(f):\ ||f-X^0||_\alpha\geq\eta\}$ has compact level sets, the "$\inf\{I_{X_0}(f)\ :\ ||f-X^0||_\alpha\geq\eta\}$" is obtained at some function f_0. Because $I_{X_0}(f)=0$ if and only if $f=X^0_{X_0}$, we conclude that

$$-\inf\{I_{X_0}(f)\ :\ ||f-X^0||_\alpha\geq\eta\}<0.$$

For $h(\varepsilon)\to\infty$, $\sqrt{\varepsilon}h(\varepsilon)\to 0$, we have

$$\limsup_{\varepsilon\to 0}h^{-2}(\varepsilon)\log\mathbb{P}\big(||X^\varepsilon-X^0||^T_\infty\geq\eta\big)=-\infty. \tag{3.19}$$

Since $\eta>0$ is arbitrary, putting together (3.17), (3.18) and (3.19), we obtain

$$\limsup_{\varepsilon\to 0}h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{||k^\varepsilon_3||_\alpha}{h(\varepsilon)}\geq\delta\bigg)=-\infty. \tag{3.20}$$

Step 2. For the first term $k^\varepsilon_1(t)$, let

$$k^\varepsilon_1(t,x)=\int_0^t\int_D\Delta G_{t-s}(x,y)\mathfrak{B}^\varepsilon(s,y)dsdy,$$

where

$$\mathfrak{B}^\varepsilon(s,y)=\bigg(\frac{f(X^\varepsilon(s,y))-f(X^0(s,y))}{\sqrt{\varepsilon}}-f'(X^0(s,y))\eta^\varepsilon(s,y)\bigg),$$

as stated in the proof of Theorem 2, we have

$$||\mathfrak{B}^\varepsilon||^T_\infty\leq C\frac{(||X^\varepsilon_{X_0}-X^0_{X_0}||^T_\infty)^2}{\sqrt{\varepsilon}}.$$

However, by Hölder's continuity of Green function G, it is easy to prove that, for any $\alpha\in(0,\frac{1}{4})$

$$|k^\varepsilon_2|^T_\alpha\leq C(\alpha,T)||\mathfrak{B}^\varepsilon||^T_\infty.$$

From the proof of proposition 1, we obtain that

$$||X^\varepsilon_{X_0}-X^0_{X_0}||^T_\infty\leq C(K_b,T)||\widetilde{k}^\varepsilon_2||^T_\infty$$

where

$$\widetilde{k}^\varepsilon_2(t,x)=\bigg(\varepsilon\int_0^t\int_D\Delta G_{t-s}(x,y)\sigma(X^\varepsilon_{X_0}(s,y))W(dsdy)\bigg)^{\frac{1}{2}}.$$

Applying lemma 3, we have

$$F(t,x,u,z)=G_{t-u}(x,z)1_{[u\leq t]},\alpha_0=\frac{1}{2},C_F=C,\rho=\sqrt{\varepsilon}K(1+||X^T_{X_0}||^T_\infty+\eta)$$

$$\widetilde{Z}(t,x)=\sqrt{\varepsilon}\sigma(X^\varepsilon_{X_0}(t,x))1_{[||X^\varepsilon_{X_0}||^T_\infty<||X^0_{X_0}||^T_\infty+\eta]},$$

for any $\eta>0$, we obtain that for all ε is sufficiently small such that
$M\geq\sqrt{\varepsilon}K(1+||X^T_{X_0}||^T_\infty+\eta)CC(\alpha,\frac{1}{2})$,

$$\begin{aligned}&\mathbb{P}(||\widetilde{k}^\varepsilon_2||^T_\infty\geq M,||X^\varepsilon_{X_0}||^T_\infty<||X^0_{X_0}||^T_\infty+\eta)\\ &\leq(\sqrt{2}T^2+1)\exp\bigg(-\frac{M^2}{\varepsilon K^2CC^2(\alpha,\frac{1}{2})(1+||X^0_{X_0}||^T_\infty+\eta)^2}\bigg).\end{aligned}$$

For the same reason as (3.20), we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}||^{T}_{\infty} \geq ||X^{0}_{X_0}||^{T}_{\infty}+\eta)\\
&\qquad\leq \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}-X^{0}_{X_0}||^{T}_{\infty}\geq\eta)\\
&\qquad= -\infty.
\end{aligned}$$

For any $\eta > 0$, by Bernstein's inequality and the continuity of σ, we have

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\bigg(\frac{|k^{\varepsilon}_1(t)|^{T}_{\alpha}}{h(\varepsilon)}\geq\delta\bigg)$$

$$\begin{aligned}
\leq\ & \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\Bigg(\bigg(||\widetilde{k}^{\varepsilon}_2||^{T}_{\infty}\bigg)^2\geq\frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,T,K_f,C)}\Bigg)\\
\leq\ & \limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\Bigg[\mathbb{P}\bigg(\big(||\widetilde{k}^{\varepsilon}_2(t)||^{T}_{\infty}\big)^2\geq\frac{\sqrt{\varepsilon}h(\varepsilon)\delta}{C(\alpha,T,K_f,C)}\ ,\\
& ||X^{\varepsilon}_{X_0}|| < ||X^{0}_{X_0}||^{T}_{\infty}+\eta\bigg)+\mathbb{P}(||X^{\varepsilon}_{X_0}||\geq||X^{0}_{X_0}||^{T}_{\infty}+\eta)\Bigg]\\
\leq\ & \Bigg(\limsup_{\varepsilon\to 0}\frac{-\delta}{\sqrt{\varepsilon}h(\varepsilon)C(\alpha,T,K_f,C)K^2CC^2(\alpha,\frac{1}{2})(1+||X_{X_0}||^{T}_{\infty}+\eta)^2}\Bigg)\\
\vee & \Bigg(\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}(||X^{\varepsilon}_{X_0}||\geq||X^{0}_{X_0}||^{T}_{\infty}+\eta)\Bigg) = -\infty. \qquad \square
\end{aligned}$$

4 A FEW EXAMPLES

4.1 Example one. Central limit theorem for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients

Let $\mathcal{O}$ be an open connected set in $\mathbb{R}^3$ such that $\mathcal{O} = [0,\pi]^3$ and $\mathcal{C}^{\alpha}([0,1]\times\mathcal{O})$ denotes the set of α-Hölder continuous fonctions. Let $\{u^{\varepsilon}(t,x)\}_{\varepsilon>0}$ be the solution of stochastic Cahn-Hilliard equation indexed by $\varepsilon > 0$, given by

$$\begin{cases}
\partial_t u^{\varepsilon}(t,x) = -\Delta\big(\Delta u^{\varepsilon}(t,x) - 4(u^{\varepsilon}(t,x))^3 + 4u^{\varepsilon}(t,x)\big) + \sqrt{\varepsilon}(1-u^{\varepsilon}(t,x))\dot{W},\\
\frac{\partial u^{\varepsilon}(t,x)}{\partial\nu} = \frac{\partial\Delta u^{\varepsilon}(t,x)}{\partial\nu} = 0, \text{ on } \quad (t,x)\in[0,T]\times\partial\mathcal{O}\\
u^{\varepsilon}(0,x) = u_0(x)
\end{cases}\tag{4.1}$$

where the coefficients f and σ are bounded, uniformly Lipschitz and verify the condition (1.2) and (1.3) such that $K_f = 16$ and $K_\sigma = 1$. Consider the process $\beta^{\varepsilon}(t,x)$ such that

$$\beta^{\varepsilon}(t,x) = \bigg(\frac{u^{\varepsilon}-u^0}{\sqrt{\varepsilon}}\bigg)(t,x).\tag{4.2}$$

In this section, we establish the CLT for the stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficients in Hölder norm $||.||_{\alpha}$ such that for all $u : [0,1]\times\mathcal{O}\longrightarrow\mathbb{R}$,

$$||u||_{\alpha} = \sup_{(s,x)\in[0,T]\times\mathcal{O}}|u(s,x)| + \sup_{\substack{(s_1,x_1)\in[0,T]\times\mathcal{O}\\(s_2,x_2)\in[0,T]\times\mathcal{O}}}\frac{|u(s_1,x_1)-u(s_2,x_2)|}{(|s_1-s_2|+|x_1-x_2|^2)^{\alpha}}.$$

Now, we obtain the main results similary to Theorem 2.

Theorem 5: *For any $\alpha\in[0,\frac{1}{4})$, $r\geq 1$, the process $\beta^{\varepsilon}(t,x)$ defined by (4.2) converges in L^r to the random process $\beta^0(t,x)$ as $\varepsilon\to 0$ where $\beta^0(t,x)$ verifies the stochastic partial differential equation*

$$\partial_t\beta^0(t,x) = -\Delta(\Delta\beta^0(t,x) - 4\big(3(u^0(t,x))^2-1)\beta^0(t,x)\big) + (1-u^0(t,x))\dot{W}(t,x)$$

with initial condition $\eta^0(0,x)=0$.

Proof of Theorem 5 : Consider the process $\beta^\varepsilon(t,x)$ defined by (4.2) depending on $u^\varepsilon(t,x)$ and $u^0(t,x)$ such that

$$\begin{aligned}&\beta^\varepsilon(t,x)\\ &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\Big(\frac{(u^\varepsilon(s,y))^3-u^\varepsilon(s,y)-((u^0(s,y))^3-u^0(s,y))}{\sqrt{\varepsilon}}\Big)dsdy\\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

Using the equality $\forall a,b\neq 0,\frac{a^3-b^3}{a-b}=a^2+ab+b^2$, we obtain

$$\begin{aligned}\beta^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\big[(u^\varepsilon(s,y))^2+u^\varepsilon(s,y).u^0(s,y)\\ &+ (u^0(s,y))^2-1\big]\beta^\varepsilon(s,y)dsdy\\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^\varepsilon(s,y))W(ds,dy)\end{aligned}$$

For $\varepsilon\to 0$, we obtain

$$\begin{aligned}\beta^0(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta_{t-s}G(x,y)\big(3(u^0(s,y))^2-1\big)\beta^0(s,y)dsdy\\ &+ \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(1-u^0(s,y))W(ds,dy).\end{aligned}$$

Denote the process $\mathcal{R}^\varepsilon=\beta^\varepsilon-\beta^0$ such that

$$\mathcal{R}^\varepsilon = m_1^\varepsilon(t,x)+m_2^\varepsilon(t,x)+m_3^\varepsilon(t,x)$$

where

$$\begin{aligned}m_1^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\Big[\Big(\frac{(u^\varepsilon(s,y))^3-(u^0(s,y))^3}{\sqrt{\varepsilon}}\Big)\\ &\quad -\Big(\frac{u^\varepsilon(s,y)-u^0(s,y)}{\sqrt{\varepsilon}}\Big)-\big(3(u^0(s,y))^2-1\big)\beta^\varepsilon(s,y)\Big]dsdy,\\ m_2^\varepsilon(t,x) &= 4\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\big(3(u^0(s,y))^2-1\big)\big(\beta^\varepsilon(s,y)-\beta^0(s,y)\big)dsdy,\\ m_3^\varepsilon(t,x) &= \int_0^t\int_{\mathcal{O}}\Big(\sum_{i=0}^{\infty}e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\Big)(u^0(s,y)-u^\varepsilon(s,y))W(ds,dy).\end{aligned}$$

Step 1. For $p>2$ and $t\in[0,1]$, we obtain

$$\begin{aligned}\mathbb{E}\big(\big|\big|m_3^\varepsilon(t,x)\big|\big|_\infty^t\big) &\leq C(p,T)\int_0^t\mathbb{E}\big(||u^\varepsilon-u^0||_\infty^s\big)^p ds\\ &\leq \sqrt{\varepsilon}C(p,T,u_0).\end{aligned}$$

By Taylor's formula, there exists a random field $\gamma^\varepsilon(t,x)$ taking values in $[0,1]$ such that

$$\begin{aligned}&f(u^\varepsilon(s,y))-f(u^0(s,y))\\ &= f'\big(u^0(s,y)+\beta^\varepsilon(t,x)(u^\varepsilon(s,y)-u^0(s,y))\big)(u^\varepsilon(s,y)-u^0(s,y))\end{aligned}$$

For the first term $m_1^\varepsilon(t,x)$, we have

$$\big|m_1^\varepsilon(t,x)\big| \leq 4\sqrt{\varepsilon}C\int_0^t\int_{\mathcal{O}}\Delta G_{t-s}(x,y)\big(\beta^\varepsilon(s,y)\big)^2dsdy. \tag{4.3}$$

By Hölder's inequality, for $p>2$

$$\mathbb{E}\big(\big|m_1^\varepsilon(t,x)\big|_\infty^t\big)^p$$
$$\leq (\sqrt{\varepsilon})^pC^p\Big(\sup_{0\leq s\leq T\ ,\ x\in\mathcal{O}}\Big|\int_0^t\int_{\mathcal{O}}\Delta G_s^q(x,y)dsdy\Big|\Big)^{\frac{p}{q}}\times\int_0^t\mathbb{E}\big(||\beta^\varepsilon||_\infty^s\big)^{2p}ds$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using (1.5) and applying proposition 1, there exists a constant $\aleph_{p,K,C}$ depending on p, K, C such that

$$\mathbb{E}\big|m_1^\varepsilon(t,x)\big|^p \leq \sqrt{\varepsilon}.\aleph_{p,K,C}. \tag{4.4}$$

Since $|f^{'}|\leq 16$,by Hölder inequality , we deduce that for $p>2$

$$\begin{aligned}\mathbb{E}|m_2^\varepsilon(t,x)|^p &\leq 2^{4p}\Big(\sup_{0\leq s\leq T\ ,x\in}\Big|\int_0^t\int_{\mathcal{O}}\Delta G_s^q(x,y)dsdy\Big|\Big)^{\frac{p}{q}}\\ &\quad\times\int_0^t\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^pds\end{aligned} \tag{4.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Putting (4.3),(4.4) and (4.5) together, we have

$$\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p \leq \aleph_{p,K,C}\big(\sqrt{\varepsilon}+\int_0^t\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^pds\big).$$

By Gronwall's inequality, we obtain

$$\mathbb{E}\big(||\beta^\varepsilon-\beta^0||_\infty^s\big)^p \leq \sqrt{\varepsilon}\aleph_{p,K,C}\to 0\ \text{ for }\ \varepsilon\to 0.$$

Step 2. We prove that the terms k_i^ε , $i=1,2,3$ satisfy the condition (A-2) in Lemma 2.
For any $p>2$ and $q^{'}\in(1,\frac{3}{2})$such that $\gamma:=(3-2q^{'})p/(4q^{'})-2>0$, for all $x,y\in\mathcal{O}$, $0\leq t\leq T$,by Burkholder's inequality and Hölder's inequality, we have

$$\mathbb{E}\big|m_3^\varepsilon(t,x)-m_3^\varepsilon(t,y)\big|^p \leq C(p,q^{'},K,T)|x-y|^{\frac{(3-2q^{'})p}{2q^{'}}} \tag{4.6}$$

where (1.3), 4 in Lemma 1 and Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$.
Similarly, in view of 5, 6 in Lemma 1; its follows that for $0\leq s\leq t\leq T$, we have

$$\mathbb{E}\big|m_3^\varepsilon(t,y)-m_3^\varepsilon(s,y)\big|^p \leq C(p,q^{'},K,T)|t-s|^{\frac{(3-2q^{'})p}{4q^{'}}} \tag{4.7}$$

where Proposition 1 were used, $\frac{1}{p^{'}}+\frac{1}{q^{'}}=1$, $C(p,q^{'},K,T)$ is independent of ε .
Putting together (4.6) and (4.7), we have

$$\mathbb{E}\big|m_3^\varepsilon(t,x)-m_3^\varepsilon(s,y)\big|^p\leq C(p,q^{'},K_\sigma,K,T)\big(|t-s|+|x-y|^2\big)^\gamma. \tag{4.8}$$

Consequently, from 4, 6 in Lemma 1, proposition 1 and the result of step 1, we also have :

$$\mathbb{E}\big|m_i^\varepsilon(t,x)-m_i^\varepsilon(s,y)\big|^p\leq C\big(|t-s|+|x-y|^2\big)^\gamma\ \ ,\ \ i=2,3. \tag{4.9}$$

Putting together (4.8) and (4.9), we obtain that there exists a constant C independent of ε satisfying that

$$\mathbb{E}\big|(\beta^\varepsilon(t,x)-\beta^0(t,x))-(\beta^\varepsilon(s,y)-\beta^0(s,y))|^p\leq C\big(|t-s|+|x-y|^2\big)^\gamma.$$

For any $\alpha\in(0,\frac{1}{4})$, $r\geq 1$, choosing $p>2$, and $q^{'}\in(1,\frac{3}{2})$ such that $\alpha\in(0,\frac{\gamma}{p})$ and $r\in[1,p)$, Lemma 2 we have

$$\lim_{\varepsilon\to 0}\mathbb{E}||\beta^\varepsilon-\beta||_\alpha^r=0.$$

4.2 Example two. Moderate Deviations Principle for stochastic Cahn-Hilliard equation with uniformly Lipschitzian coefficient

In this section we establish the MDP for the stochastic Cahn-Hilliard equation (4.1). Consider the process $\Theta^\varepsilon(t,x)$ such that

$$\Theta^\varepsilon(t,x) := \left(\frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}a(\varepsilon)}\right)(t,x). \tag{4.10}$$

In this section, we study the LDP for $\Theta^\varepsilon(t,x)$ defined by (4.10) as $\varepsilon \to 0$ with $1 < a(\varepsilon) < \frac{1}{\sqrt{\varepsilon}}$.

Theorem 6: *The process $\{\Theta^\varepsilon(t,x)\}_{\varepsilon>0}$ defined by (4.10) obeys a LDP on the space $\mathcal{C}^\alpha([0,1]\times\mathcal{O})$, with speed $a^2(\varepsilon)$ and rate function $\mathcal{J}_{M.D.P}(.)$ such that :*

$$\mathcal{J}_{M.D.P}(g) = \inf_{g=\mathcal{G}^0(u_0,\mathcal{I}(h))}\left\{\frac{1}{2}\int_0^T\int_0^\pi\int_0^\pi\int_0^\pi \dot{h}^2(t,x)dtdx_1dx_2dx_3\right\}$$

and $+\infty$ otherwise.

Proof of Theorem 6: It is sufficient to prove that

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon) \quad \log \quad \mathbb{P}\left(\frac{|\beta^\varepsilon - \beta^0|_\alpha}{a(\varepsilon)} > \delta\right) = -\infty \quad , \quad \forall\delta > 0.$$

Recall the decomposition in the proof of Theorem 5

$$\beta^\varepsilon(t,x) - \beta^0(t,x) = m_1^\varepsilon(t,x) + m_2^\varepsilon(t,x) + m_2^\varepsilon(t,x).$$

For any q in $(\frac{3}{2},3)$, $\frac{1}{p}+\frac{1}{q}=1$, and $x,y\in\mathcal{O}$, $0\le s\le t\le T$, by Hölder's inequality, 4 in Lemma 1 and (2.3), we have

$$\left|m_2^\varepsilon(t,x) - m_2^\varepsilon(t,y)\right|^p \quad \le \quad 16|x-y|^{\frac{3-q}{q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}. \tag{4.11}$$

Similarly, in view of 5 and 6, it follows that for $0\le s\le t\le T$,

$$\left|m_2^\varepsilon(t,y) - m_2^\varepsilon(s,y)\right|^p \quad \le \quad 32|t-s|^{\frac{3-q}{2q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}. \tag{4.12}$$

Putting together (4.11), (4.12), we have

$$\left|m_2^\varepsilon(t,y) - m_2^\varepsilon(s,y)\right|^p \le C(K_f)(|t-s| + |x-y|^2)^{\frac{3-q}{2q}} \times \left(\int_0^t (||\beta^\varepsilon - \beta^0||_\infty^u)^p du\right)^{\frac{1}{p}}.$$

Choosing $q\in(\frac{3}{2},3)$, such that $\alpha = 3-q/2q$ and noticing that $||\beta^\varepsilon - \beta^0||_\infty^u \le (1+u)^\alpha|\beta^\varepsilon - \beta^0|_\alpha^u$, we obtain that

$$|m_2^\varepsilon|_\alpha^t \le C(K_f)\left(\int_0^t ((1+u)^\alpha|\beta^\varepsilon - \beta^0|_\alpha^u)^p du\right)^{\frac{1}{p}}.$$

Thus, for $t\in[0,1]$, we have

$$(|\beta_t^\varepsilon - \beta_t^0|_\alpha^t)^p \le C(p,T,K_f)\left[\left(|m_1^\varepsilon(t)|_\alpha^t + |m_3^\varepsilon(t)|_\alpha^t\right)^p + \int_0^t (|\beta^\varepsilon - \beta^0|_\alpha^s)^p ds\right].$$

Applying Gronwall's Lemma to $\Psi(t) = (|\beta_t^\varepsilon - \beta_t^0|_\alpha^t)^p$, we have

$$(|\beta_t^\varepsilon - \beta_t^0|_\alpha^t)^p \le C(p,T,K_f)\left[\left(|m_1^\varepsilon(t)|_\alpha^t + |m_3^\varepsilon(t)|_\alpha^t\right)^p\right]e^{C(p,T,K_f)T}. \tag{4.13}$$

By (4.12) and (4.13), it is sufficient to prove that for any $\delta > 0$,

$$\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log\mathbb{P}\left(\frac{|m_i^\varepsilon(t)|_\alpha^T}{a(\varepsilon)} > \delta\right) = -\infty \qquad\qquad i = 1,3.$$

Step 1. For any $\varepsilon > 0, \eta > 0$ we have

$$\begin{aligned}\mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta\big) &\leq \mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta, |u^\varepsilon - u^0|_\infty^T < \eta\big) \\ &+ \mathbb{P}(|u^\varepsilon - u^0|_\infty^T \geq \eta)\end{aligned} \tag{4.14}$$

By 4 and 6 in Lemma 1, $\big(\sum_{i=0}^\infty e^{-\mu_i^2(t-s)}w_i(x)w_i(y)\big).1_{[u\leq t]}$ satisfies (3.12)(see Lemma 3) for $\alpha_0 = \frac{1}{2}$. Applying Lemma 3, we have

$$F(t,x,u,z) = \big(\sum_{i=0}^\infty e^{-\mu_i^2(t-s)}w_i(x)w_i(z)\big)1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, M = a(\varepsilon)\delta,$$

$\rho = \eta K_\sigma, Y^*(t,x) = \big(u^0(t,x) - u^\varepsilon(t,x)\big)1_{||u^\varepsilon - u^0||_\infty^T > \eta}$
we obtain that for all ε sufficiently small such that $a(\varepsilon)\delta \geq \rho C C(\alpha, \frac{1}{2})$

$$\mathbb{P}\big(|m_3^\varepsilon(t)|_\alpha^T > a(\varepsilon)\delta, ||u^\varepsilon - u^0||_\infty^T < \eta\big) \leq (\sqrt{2}T^2 + 1)\exp\big(-\frac{a^2(\varepsilon)\delta^2}{\eta^2 K_\sigma^2 C C^2(\alpha, \frac{1}{2})}\big). \tag{4.15}$$

Since u^ε satisfies the LDP on $\mathcal{C}^\alpha([0,T]\times\mathcal{O})$

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(||u^\varepsilon - u^0||_\infty^T \geq \eta) &\leq \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(||u^\varepsilon - u^0||_\alpha \geq \eta) \\ &\leq -\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}.\end{aligned}$$

In this case, the good rate function $\mathcal{I} = \{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}$ has compact level sets, the "$\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\}$" is obtained at some function f_0. Because $\mathcal{I}(f) = 0$ if and only if $f = u^0$, we conclude that

$$-\inf\{\mathcal{I}(f) \ : \ ||f - u^0||_\alpha \geq \eta\} < 0.$$

For $a(\varepsilon) \to \infty$, $\sqrt{\varepsilon}a(\varepsilon) \to 0$, we have

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log\mathbb{P}\big(||u^\varepsilon - u^0||_\infty^T \geq \eta\big) = -\infty. \tag{4.16}$$

Since $\eta > 0$ is arbitrary, putting together (4.14), (4.15) and (4.16), we obtain

$$\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log\mathbb{P}\left(\frac{||m_3^\varepsilon||_\alpha}{a(\varepsilon)} \geq \delta\right) = -\infty. \tag{4.17}$$

Step 2. For the first term $m_1^\varepsilon(t)$, let

$$m_1^\varepsilon(t,x) = \int_0^t \int_{\mathcal{O}} \Delta G_{t-s}(x,y)\mathfrak{M}^\varepsilon(s,y)dsdy,$$

where

$$\begin{aligned}\mathfrak{M}^\varepsilon(s,y) &= 4 \quad \Bigg(\bigg(\frac{(u^\varepsilon(s,y))^3 - (u^0(s,y))^3}{\sqrt{\varepsilon}}\bigg) - \bigg(\frac{u^\varepsilon(s,y) - u^0(s,y)}{\sqrt{\varepsilon}}\bigg) \\ &\quad -\big(3(u^0(s,y))^2 - 1\big)\beta^\varepsilon(s,y)\Bigg)\end{aligned}$$

as stated in the proof of Theorem 5, we have

$$||\mathfrak{M}^\varepsilon||_\infty^T \leq C\frac{(||u^\varepsilon - u^0||_\infty^T)^2}{\sqrt{\varepsilon}}.$$

However, by the Hölder's continuity of Green function G, it is easy to prove that, for any $\alpha \in (0, \frac{1}{4})$

$$|m_2^\varepsilon|_\alpha^T \leq C(\alpha, T)||\mathfrak{M}^\varepsilon||_\infty^T.$$

From the proof of proposition 1, we obtain that

$$||u^\varepsilon - u^0||_\infty^T \leq C(T)||\widetilde{m}_2^\varepsilon||_\infty^T.$$

where

$$\widetilde{m}_2^\varepsilon(t,x) = \sqrt{\varepsilon \int_0^t \int_{\mathcal{O}} \Delta G_{t-s}(x,y) u^\varepsilon(s,y) W(dsdy)}.$$

Applying lemma 3, we have

$$F(t,x,u,z) = G_{t-u}(x,z)1_{[u\leq t]}, \alpha_0 = \frac{1}{2}, C_F = C, \rho = \sqrt{\varepsilon}K(1+||u^T||_\infty^T + \eta)$$

$$Z^*(t,x) = \sqrt{\varepsilon}(1-u^\varepsilon(t,x))1_{[||u^\varepsilon||_\infty^T < ||u^0||_\infty^T + \eta]},$$

for any $\eta > 0$, we obtain that for all ε is sufficiently small such that $M \geq \sqrt{\varepsilon}(1+||u^T||_\infty^T + \eta)CC(\alpha, \frac{1}{2})$,

$$\begin{aligned}
&\mathbb{P}(||\widetilde{m}_2^\varepsilon||_\infty^T \geq M, ||u^\varepsilon||_\infty^T < ||u^0||_\infty^T + \eta) \\
&\leq (\sqrt{2}T^2+1)\exp\bigg(-\frac{M^2}{\varepsilon K^2 CC^2(\alpha,\frac{1}{2})(1+||u^0||_\infty^T + \eta)^2}\bigg).
\end{aligned}$$

For the same raison as (4.11), we obtain

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log \mathbb{P}(||u^\varepsilon||_\infty^T \geq ||u^0||_\infty^T + \eta) \\
\leq\ &\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log \mathbb{P}(||u^\varepsilon - u^0||_\infty^T \geq \eta) = -\infty.
\end{aligned}$$

For any $\eta > 0$, by Bernstein's inequality and the continuity of σ, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log \mathbb{P}\bigg(\frac{|m_1^\varepsilon(t)|_\alpha^T}{a(\varepsilon)} \geq \delta\bigg) \\
\leq\ &\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log \mathbb{P}\bigg(\bigg(||\widetilde{m}_2^\varepsilon||_\infty^T\bigg)^2 \geq \frac{\sqrt{\varepsilon}a(\varepsilon)\delta}{C(\alpha,T,K_f,C)}\bigg) \\
\leq\ &\limsup_{\varepsilon\to 0} a^{-2}(\varepsilon)\log \bigg[\mathbb{P}\bigg(\big(||\widetilde{m}_2^\varepsilon(t)||_\infty^T\big)^2 \geq \frac{\sqrt{\varepsilon}a(\varepsilon)\delta}{C(\alpha,T,K_f,C)}, \\
&||u^\varepsilon|| < ||u^0||_\infty^T + \eta\bigg) + \mathbb{P}(||u^\varepsilon|| \geq ||u^0||_\infty^T + \eta)\bigg] \\
\leq\ &\bigg(\limsup_{\varepsilon\to 0} \frac{-\delta}{\sqrt{\varepsilon}a(\varepsilon)C(\alpha,T,K_f,C)K^2CC^2(\alpha,\frac{1}{2})(1+||u^0||_\infty^T + \eta)^2}\bigg) \\
&\vee\bigg(\limsup_{\varepsilon\to 0} h^{-2}(\varepsilon)\log \mathbb{P}(||X_{X_0}^\varepsilon|| \geq ||X_{X_0}^0||_\infty^T + \eta)\bigg) = -\infty.
\end{aligned}$$

5 Conclusion

In this paper, we have proved a CLT and a MDP for a perturbed stochastic Cahn-Hilliard equation in Hölder space by using the exponential estimates in the space of Hölder continuous functions and the Garsia-Rodemich-Rumsey's lemma. We can also examine the same situation in Besov-Orlicz space.

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